

THE HEIGHT OF THE AUTOMORPHISM TOWER OF A GROUP

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ABSTRACT. For a group G with trivial center there is a natural embedding of G into its automorphism group, so we can look at the latter as an extension of the group. So an increasing continuous sequence of groups, the automorphism tower, is defined, the height is the ordinal where this becomes fixed, arriving to a complete group. We show that for many such κ there is a group of height $> 2^\kappa$, so proving that the upper bound essentially cannot be improved.

I would like to thank Alice Leonhardt for the beautiful typing.
Research partially supported by NSF Grant No. nsf-dms 0100794. Publication 810

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\text{\textsf{TEX}}$

§0 INTRODUCTION

For a group G with trivial center there is a natural embedding of G into its automorphism group $\text{Aut}(G)$ where $g \in G$ is mapped to the inner automorphism $x \mapsto gxg^{-1}$ which is defined and is not the identity for $g \neq e_G$ as G has a trivial center, so we can view $\text{Aut}(G)$ as a group extending G . Also the extension $\text{Aut}(G)$ is a group with trivial center, so we can continue defining $G^{<\alpha>}$ increasing with α for every ordinal α ; let τ_G be when we stop, i.e., the first α such that $G^{<\alpha+1>} = G^{<\alpha>}$ (or $\alpha = \infty$ but see below) hence $\beta > \alpha \Rightarrow G^{<\beta>} = G^{<\alpha>}$, (see Definition 0.2). How large can τ_G be?

Weilant [Wel39] proves that for finite G , τ_G is finite. Thomas [Th85] celebrated work proves for infinite G that $\tau_G \leq (2^{|G|})^+$, in fact as noted by Felgner and Thomas $\tau_G < (2^{|G|})^+$. Thomas shows also that $\tau_\kappa \geq \kappa^+$. Later he ([Th98]) showed that if $\kappa = \kappa^{<\kappa>}$, $2^\kappa = \kappa^+$ (hence $\tau_\kappa < \kappa^{++}$ in \mathbf{V}) and $\lambda \geq \kappa^{++}$ and we force by \mathbb{P} , the forcing of adding λ Cohen subsets to κ , then in $\mathbf{V}^\mathbb{P}$ we still have $\tau_\kappa < \kappa^{++}$ though 2^κ is $\geq \lambda$ (and $\mathbf{V}, \mathbf{V}^\mathbb{P}$ has the same cardinals).

Just Shelah and Thomas [JShT 654] prove that when $\kappa = \kappa^{<\kappa>} < \lambda$, in some forcing extension (by a specially constructed κ -complete κ^+ -c.c. forcing notion) we have $\tau_\kappa \geq \lambda$, so consistently $\tau_\kappa > 2^\kappa > \kappa^+$ for some κ . An important lemma there which we shall use (see 0.6 below) is that if G is the automorphism group of a structure of cardinality κ , $H \subseteq G$, $|H| \leq \kappa$ then $\tau'_{G,H}$, the normalizer length of H in G (see Definition 0.3(2)), is $< \tau_\kappa$. Concerning groups with center Hamkins show that $\tau_G < \text{the first strongly inaccessible cardinal} > |G|$. On the subject see the forthcoming book of Thomas.

We shall show, e.g.

0.1 Theorem. *If κ is strong limit singular of uncountable cofinality then $\tau_\kappa > 2^\kappa$.*

It would have been nice if the lower bound for τ_κ, κ^+ would (consistently) be the correct one, but Theorem 0.1 shows that this is not so. Note that Theorem 0.1 shows that provably in ZFC, in general the upper bound $(2^\kappa)^+$ cannot be improved. See Conclusion 3.12 for proof of the theorem, quoting results from pcf theory. We thank Simon Thomas, the referee and Itay Kaplan for many valuable complaints detecting serious problems in earlier versions.

The program, described in a simplified way, is that for each so called “ κ -parameter \mathbf{p} ” which includes a partial order I we define a group $G_\mathbf{p}$ and a two element subgroup $H_\mathbf{p}$ such that $\langle \text{nor}_{G_\mathbf{p}}^\alpha(H_\mathbf{p}) : \alpha \leq \text{rk}_I^{<\infty} \rangle$ “reflect” $\text{rk}_I^{<\infty} = \text{rk}_\mathbf{p}$, the natural rank on I (see Definition 1.1), so in particular $\tau'_{G_\mathbf{p}, H_\mathbf{p}} = \text{rk}_\mathbf{p}^{<\infty}$. (Actually in the end we shall get only H of cardinality $\leq \kappa$).

We use an inverse system $\mathfrak{s} = \langle \mathbf{p}_u, \pi_{u,v} : u \leq_J v \rangle$ of κ -parameters $\pi_{u,v}$ maps $I_{\mathbf{p}_v}$ to $I_{\mathbf{p}_u}$; however, in general the $\pi_{u,v}$'s do not preserve order (but do preserve in some weak global sense) where J is an \aleph_1 -directed partial order. Now for each $u \in J$, we can define the group $G_{\mathbf{p}_u}$; and we can take inverse limit in two ways.

Way 1: The inverse limit $\mathbf{p}_\mathfrak{s}$ (with $\pi_{u,\mathfrak{s}}$ for $u \in J$ of \mathfrak{s}) is a κ -parameter and so the group $G_{\mathbf{p}_\mathfrak{s}}$ is well defined.

Way 2: The inverse system $\langle G_{\mathbf{p}_u}, \hat{\pi}_{u,v} : u \leq_J v \rangle$, of groups where $\hat{\pi}_{u,v}$ is the (partial) homomorphism from $G_{\mathbf{p}_v}$ to $G_{\mathbf{p}_u}$ induced by $\pi_{u,v}$, has an inverse limit $G_\mathfrak{s}$.

Now

- (A) concerning $G_{I_\mathfrak{s}}$ we normally have good control over $\text{rk}(\mathbf{p}_\mathfrak{s})$ hence on the normalizer length of $H_{\mathbf{p}_\mathfrak{s}}$ inside $G_{\mathbf{p}_\mathfrak{s}}$
- (B) $G_\mathfrak{s}$ is (more exactly can be represented as good enough) inverse limit of groups of cardinality $\leq \kappa$ hence is isomorphic to $\text{Aug}(\mathfrak{A})$ for some structure of cardinality $\leq \kappa$
- (C) in the good case $G_{\mathbf{p}_\mathfrak{s}} = G_\mathfrak{s}$ so we are done (by 0.6).

In §3 we work to get the main result.

There are obvious possible improvement of the results here, say trying to prove $\delta_\kappa \leq \tau_\kappa$ (see Definition 0.5) for every κ . But more importantly, a natural conjecture, at least for me was $\tau_\kappa = \delta_\kappa$ because all the results so far on τ_κ has parallel for δ_κ (though not inversely). In particular it seems reasonable that for $\kappa = \aleph_0$ the lower bound was right, i.e., $\tau_\kappa = \omega_1$. [We shall try to return to those problems in a sequel [Sh:F579].]

0.2 Definition. 1) For a group G with trivial center, define the group $G^{<\alpha>}$ with trivial center for an ordinal α , increasing continuous with α such that $G^{<0>} = G$ and $G^{<\alpha+1>}$ is the group of automorphisms of $G^{<\alpha>}$ identifying $g \in G^{<\alpha>}$ with the inner automorphisms it defines. We may stipulate $G^{<-1>} = \{e_G\}$.

[We know that $G^{<\alpha>}$ is a group with trivial center increasing continuous with α and for some $\alpha < (2^{|G|+\aleph_0})^+$ we have $\beta > \alpha \Rightarrow G^{<\beta>} = G^{<\alpha>}$.]

2) The automorphism tower height of the group G is $\tau_G = \tau_G^{\text{atw}} = \text{Min}\{\alpha : G^{<\alpha>} = G^{<\alpha+1>}\}$; clearly $\beta \geq \alpha \geq \tau_G \Rightarrow G^{<\beta>} = G^{<\alpha>}$, atw stands for automorphism tower.

3) Let $\tau_\kappa = \tau_\kappa^{\text{atw}}$ be the least ordinal τ such that $\tau(G) < \tau$ for every group G of cardinality $\leq \kappa$; we call it the group tower ordinal of κ .

Now we define normalizer (group theorist write $N_G(H)$, but probably for others $\text{nor}_G(H)$ will be clearer, at least this is so for the author).

0.3 Definition. 1) Let H be a subgroup of G .

We define $\text{nor}_G^\alpha(H)$, a subgroup of G , by induction on the ordinal α , increasing continuous with α . We may add $\text{nor}_G^{-1}(H) = \{e_G\}$.

Case 1: $\alpha = 0$.

$$\text{nor}_G^0(H) = H.$$

Case 2: $\alpha = \beta + 1$.

$$\text{nor}_G^\alpha(H) = \text{nor}_G(\text{nor}_G^\beta(H)), \text{ see below.}$$

Case 3: α a limit ordinal

$$\text{nor}_G^\alpha(H) = \cup \{ \text{nor}_G^\beta(H) : \beta < \alpha \}$$

where

$$\begin{aligned} \text{nor}_G(H) = \{ g \in G : g \text{ normalize } H, \text{ i.e. } gNg^{-1} = N, \text{ equivalently} \\ (\forall x \in H)[gxg^{-1} \in H \ \& \ g^{-1}xg \in H] \}. \end{aligned}$$

2) Let $\tau'_{G,H} = \tau_{G,H}^{\text{nlg}}$, the normalizer length of H in G , be $\text{Min}\{\alpha : \text{nor}_G^\alpha(H) = \text{nor}_G^{\alpha+1}(H)\}$; so $\beta \geq \alpha \geq \tau'_{G,H} = \text{nor}_G^\beta(H) = \text{nor}_G^\alpha(H)$; nlg stands for normalizer length.

3) Let $\tau'_\kappa = \tau_\kappa^{\text{nlg}}$ be the least ordinal τ such that $\tau > \tau'_{G,H}$ whenever $G = \text{Aut}(\mathfrak{A})$ for some structure \mathfrak{A} on κ and $H \subseteq G$ is a subgroup satisfying $|H| \leq \kappa$.

4) $\tau''_\kappa = \tau_\kappa^{\text{nlf}}$ is the least ordinal τ such that $\tau > \tau_{G,H}^{\text{nlf}}$ wherever $G = \text{Aut}(\mathfrak{A})$, \mathfrak{A} a structure of cardinality $\leq \kappa$, H a subgroup of G of cardinality $\leq \kappa$ and $\text{nor}_G^\infty(H) = \cup \{ \text{nor}_G^\alpha(H) : \alpha \text{ an ordinal} \} = G$.

0.4 Definition. We say that G is a κ -automorphism group if G is the automorphism group of some structure of cardinality $\leq \kappa$.

0.5 Definition. Let $\delta_\kappa = \delta(\kappa)$ be the first ordinal α such that there is no sentence $\psi \in \mathbb{L}_{\kappa^+, \omega}$ satisfying:

- (a) $\psi \vdash "< \text{ is a linear order}"$
- (b) for every $\beta < \alpha$ there is a model M of ψ such that $(|M|, <^M)$ has order type $\geq \beta$
- (c) for every model M of ψ , $(|M|, <^M)$ is a well ordering.

See on this, e.g. [Sh:c, VII, §5].

Our proof of better lower bounds rely on the following result from [JShT 654].

0.6 Lemma. $\tau'_\kappa \leq \tau_\kappa$.

0.7 Question: 1) Is it consistent that for some κ , $\tau'_\kappa < \tau_\kappa$? Is this provable in ZFC?

Is the negation consistent?

2) Similarly for the inequalities $\delta_\kappa < \tau'_\kappa$, (and $\delta_\kappa < \tau'_\kappa < \tau_\kappa$).

See on those in [Sh:F579].

0.8 Observation. For every $\kappa \geq \aleph_0$ we have $\tau_\kappa^{\text{atw}} \geq \tau_\kappa^{\text{nlg}} \geq \tau_\kappa^{\text{nlf}}$.

Proof. By 0.6 and checking the definitions of $\tau_\kappa^{\text{nlg}}, \tau_\kappa^{\text{nlf}}$. In fact we mostly work on proving that in 0.2, $\tau_\kappa^{\text{nlf}} > 2^\kappa$.

Notation: For a group G and $A \subseteq G$ let $\langle A \rangle_G$ be the subgroup of G generated by A .

* * *

Explanation of the proof:

We would like to derive the desired group from a partial order I representing the ordinal desired as $\tau_{G,H}$ in some way and the tower of normalizers of an appropriate subgroup will reflect. It seems natural to say that if $t \in I$ represent the ordinal α then the $s <_I t$ will represent ordinals $< \alpha$ so we use the depth in I

$$\text{dp}_I(t) = \cup \{ \text{dp}_I(s) + 1 : s <_I t \}.$$

For each $t \in I$ we will like to have a generator g_t of the group (really denoted by $g_{\langle t \rangle, \langle \cdot \rangle}$) take care of the normalizer tower not sloping at $\alpha = \text{dp}_I(t)$ say g_t will be in the $(\alpha + 1)$ -th normalizer but not in the α -th normalizer. But we need a witness for g_t not being in earlier normalizer $(\beta + 1)$ -th normalizer $\beta < \alpha$.

Now β is represented by some $s <_I t$, so we have witness $g_{\langle (t,x), \langle \cdot \rangle \rangle}, g_{\langle (t,x), (1) \rangle}$, the first in the beginning, the second in the $(\beta + 1)$ -th normalizer not in the β -th normalizer. So we have a long normalizer tower of the subgroup $G_I^{<0}$, the one generated by $\{g_{(\bar{t}, \eta)} : \eta(\ell) = 0 \text{ for some } \ell < \ell g(\eta)\}$.

However $G_I^{<0}$ is too big. So we use a semi-direct product $K_I = G_I * L_I$, where L_I is an abelian group with every element of order two, generated by $\{\mathbf{h}_{gG_I^{<0}} : g \in G_I^{<0}\}$ with $g_1 \mathbf{h}_{gG_I^{<0}} = \mathbf{h}_{(g_1 g)G_I^{<0}}$ and show that the normalizer wins of the subgroup $H_I = \{e, \mathbf{h}_{G_I^{<0}}\}$ of K_I has the same height.

But we have to make K_I a κ -automorphism group. We only almost have it: (and has too) we will represent it as $\text{aut}(M)/N$ for some structure M of cardinality $\leq \kappa$ and normal subgroup N of it of cardinality $\leq \kappa$; this suffices.

From where will M come from? We will represent I as a universe limit of some kind of $\mathbf{t} = \langle I_u, \pi_{u,v} : u \leq_J v \rangle$ where I_u is a partial order of cardinality $\leq \kappa$, $\pi_{u,v}$ a mapping from I_v to I_u (commuting). It seemed a priori natural to have $\pi_{u,v}$ is order preserving but it seemingly does not work out. It seemed a priori natural to prove that whenever \mathbf{t} is as above there is a universe limit, etc. We find it more transparent to treat axiomatically: the limit is given inside, i.e. as \mathfrak{s} which is $\mathbf{t} +$ a limit v^* ; and $J^{\mathbf{t}} = J^{\mathfrak{s}} \setminus \{v^*\}$ is directed.

Also we demand that $J^{\mathbf{t}}$ is \aleph_1 -directed (otherwise in the limit we have words come).

We shall derive the structure M from \mathbf{t} so its automorphism comes from members of K_{I_u} ($u \in J^{\mathbf{t}}$). Well, not exactly but for formal terms for it, to enable us to project to $u' \leq_{J[\mathbf{t}]} u$; as recall that $\pi_{u,v}$ does not necessarily preserve order. To make things smooth we demand that if $J^{\mathbf{t}}$ is a linear order (say $\text{cf}(\kappa)$) when as in the main case, κ is singular strong limit of uncountable cofinality.

More specifically, if $s, t \in I$ then for every large enough $u \in J^{\mathbf{t}}, s <_{I_{v^*}} t \Leftrightarrow \pi_{u,v^*}(s) <_{I_u} \pi_{u,v^*}(t)$; note the order of the quantifiers. Then we define a structure M derived from \mathbf{t} . So the automorphism group of M is the inverse limit of groups which comes from the formal definitions of elements of K_{I_u} 's. Each depend on finitely many generators, which in different u 's give different reduced forms.

Now they are defined from some $\bar{t} \in {}^k(I_u)$ using “ I_{v^*} is the inverse limit...” the “important” t_u 's, those which really affect, well form an inverse system (without loss of generality the length k is constant on an end segment here we use “ $J^{\mathbf{t}}$ is \aleph_1 -directed”) so for those ℓ 's $\langle t_{u,\ell} : u \in J^{\mathbf{t}} \rangle$ has limit $t_{v^*,\ell}$ say for $\ell < k_*$.

So $\langle t_{u^*,\ell} : \ell < k_* \rangle$ has the same quantifier type in I_u whenever $u_* \leq u \leq v^*$ for some $u_* < v^*$. The other t 's still has influence, so it is enough to find for them a pseudo limit: $t_{v^*,\ell}$ such that they will have the same affect on how the “important” $t_{u,\ell}$ are used (this is the essential limit).

All this gives an approximation to $\text{aut}(M) \cong K_{I_{v^*}}$. They almost mean that we divide by the subgroup of the automorphism of M which are id_{K_u} after $u \in J^{\mathbf{t}}$ large enough. This is a normal subgroup of cardinality $\leq \kappa$ so we are done except constructing such systems.

§1 THE GROUPS

Discussion: Our aim is for a partial order I to define a group $G = G_I$ and a subgroup $H = H_I$ such that the normalizer length of H inside G reflects the depth of the well founded part of I . Eventually we would like to use I of large depth such that $|H_I| \leq \kappa$ and the normalizer length of H inside G_I is $> \kappa$, even equal to the depth of I .

For clarity we first define an approximation, in particular, H appears only in §2. How do we define the group $G = G_I$ from the partial order I ? For each $t \in I$ we would like to have an element associated with it (it is $g_{(<t>, <>)}$) such that it will “enter” $\text{nor}_G^\alpha(H)$ exactly for $\alpha = \text{rk}_I(t) + 1$. We intend that among the generators of the group commuting is the normal case, and we need witnesses that $g_{(<t>, <>)}$ $\notin \text{nor}_G^{\beta+1}(H)$ wherever $\beta < \alpha = \text{rk}_I(t), \beta > 0$. It is natural that if $\text{rk}_I(t_1) = \beta$ and $t_1 <_I t_0 =: t$ then we use t_1 to represent β , as witness; more specifically, we construct the group such that conjugation by $g_{(<t>, <>)}$ interchange $g_{(<t_0, t_1>, <0>)}$ and $g_{(<t_0, s_0>, <1>)}$ and one of them, say $g_{(<t_0, t_1>, <0>)}$ belongs to $\text{nor}_G^{\beta+1}(H) \setminus \text{nor}_G^\beta(H)$ whereas the other one, $g_{(<t_0, 0>, <1>)}$, belongs to $\text{nor}_G^1(H)$. Iterating we get the elements $x \in X_I$ defined below.

In an earlier version, to “start the induction”, some additional generators $g_{(\alpha, \ell)}$ ($\alpha \in Z^I, \ell < 2$) were used to generate H and not using all of them had helped to make $\text{nor}_{G_I}^1(H_I)$ having the desired value. However, we have to decide for each $g_{(\bar{t}, \nu)}$ for (\bar{t}, ν) as above, for which $g_{(\alpha, \ell)}$ ($\alpha \in Z^I, \ell < 2$) does conjugation by $g_{(\bar{t}, \nu)}$ maps $g_{(\alpha, \ell)}$ to itself and for which it does not. For this we chose subsets $A_{(\bar{t}, \nu)} \subseteq Z^I$ to code our decisions when (\bar{t}, ν) is as above and well defined, and make the conjugation with the generators intended to generate $\text{nor}_G^1(H)$ appropriately.

Now we do it by adding to G an element g_* of order 2 getting K_I , commuting with $g \in G$ iff g is intended to be in the low level (e.g. $g_{(\bar{t}, \eta)}, t_n \in I$ is minimal, see notation below).

We could have in this section considered only a partial order I , and the groups G_I (and later K_I) derived from it. But as anyhow we shall use it in the context of κ -p.o.w.i.s., we do it in this frame (of course if $J^\mathfrak{s} = \{u\}$, then \mathfrak{s} is essentially just I_u).

Note that for our main result it suffices to deal with the case $\text{rk}(I) < \infty$.

1.1 Definition. Let I be a partial order (so $\neq \emptyset$).

- 1) $\text{rk}_I : I \rightarrow \text{Ord} \cup \{\infty\}$ is defined by $\text{rk}_I(t) \geq \alpha$ iff $(\forall \beta < \alpha)(\exists s <_I t)[\text{rk}_I(s) \geq \beta]$.
- 2) $\text{rk}_I^{<\infty}(t)$ is defined as $\text{rk}_I(t)$ if $\text{rk}_I(t) < \infty$ and is defined as $\cup\{\text{rk}_I(s) + 1 : s \text{ satisfies } s <_I t \text{ and } \text{rk}_I(s) < \infty\}$ in general.
- 3) Let $\text{rk}(I) = \cup\{\text{rk}_I(t) + 1 : t \in I\}$ stipulating $\alpha < \infty = \infty + 1$.
- 4) $\text{rk}_I^{<\infty} = \text{rk}^{<\infty}(I) = \cup\{\text{rk}_I^{<\infty}(t) + 1 : t \in I\}$.

- 5) Let $I_{[\alpha]} = \{t \in I : \text{rk}(t) = \alpha\}$.
 6) I is non-trivial when $\{s : s \leq_I t \text{ and } \text{rk}_I(s) \geq \beta\}$ is infinite for every $t \in I$ satisfying $\text{rk}_I^{<\infty}(t) > \beta$ (used in the proof of 1.9(1); it is equivalent to demand “ $\text{rk}_I(s) = \beta$ ”).
 7) I is explicitly non-trivial if each E_I -equivalence class is infinite where $E_I = \{(t_1, t_2) : t_2 \in I, t_2 \in I \text{ and } (\forall s \in I)(s <_I t_1 \equiv s <_I t_2)\}$.

1.2 Definition. 1) \mathfrak{s} is a κ -p.o.w.i.s. (partial order weak inverse system) when:

- (a) $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ so $J = J^{\mathfrak{s}} = J[\mathfrak{s}]$, $\bar{I} = \bar{I}^{\mathfrak{s}}$, $\bar{\pi} = \bar{\pi}^{\mathfrak{s}}$
 - (b) J is a directed partial order of cardinality $\leq \kappa$
 - (c) $\bar{I} = \langle I_u : u \in J \rangle = \langle I_u^{\mathfrak{s}} : u \in J \rangle$ and we may write $I[u]$ or $I^{\mathfrak{s}}[u]$
 - (d) $I_u = I_u^{\mathfrak{s}}$ is a partial order of cardinality $\leq \kappa$
 - (e) $\bar{\pi} = \langle \pi_{u,v} : u \leq_J v \rangle$
 - (f) $\pi_{u,v}$ is a partial mapping from I_v into I_u (no preservation of order is required!)
 - (g) if $u \leq_J v \leq_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$.
- 2) \mathfrak{s} is a p.o.w.i.s. mean κ -p.o.w.i.s. for some κ .
 3) For $u \in J$ let $X_u = X_u^{\mathfrak{s}}$ be the set of x such that for some $n < \omega$:
- (a) $x = (\bar{t}, \eta) = (\bar{t}^x, \eta^x)$
 - (b) η^x is a function from $\{0, \dots, n-1\}$ to $\{0, 1\}$
 - (c) $\bar{t} = \langle t_\ell : \ell \leq n \rangle = \langle t_\ell^x : \ell \leq n \rangle$ where $t_\ell \in I_u^{\mathfrak{s}}$ is $<_{I_u^{\mathfrak{s}}}$ -decreasing, i.e., $t_n <_{I_u^{\mathfrak{s}}} t_{n-1} <_{I_u^{\mathfrak{s}}} \dots <_{I_u^{\mathfrak{s}}} t_0$.
- 3A) In fact for every partial order I we define X_I similarly, so $X_u^{\mathfrak{s}} = X_{I^{\mathfrak{s}}[u]}$.
 4) In part (3) for $x \in X_u^{\mathfrak{s}}$ let $n(x) = \ell g(\bar{t}^x) - 1$ and $t^x = t(x) := t_{n(x)}^x$.
 5) For $x \in X_u^{\mathfrak{s}}$ and $n \leq n(x)$ let $y = x \upharpoonright n \in X_u^{\mathfrak{s}}$ be defined by:

$$\bar{t}^y := \bar{t}^x \upharpoonright (n+1) = \langle t_0^x, \dots, t_n^x \rangle$$

$$\eta^y = \eta^x \upharpoonright n(y) =: \eta^x \upharpoonright \{0, \dots, n-1\}.$$

- 6) We define $\text{rk}_u^1 = \text{rk}_u^{1,\mathfrak{s}}$ and $\text{rk}_u^2 = \text{rk}_u^{2,\mathfrak{s}}$ as follows:

- (a) let $\text{rk}_u^1 : X_u \rightarrow \text{Ord} \cup \{\infty\}$ be defined by $x \in X_u \Rightarrow \text{rk}_u^{1,\mathfrak{s}}(x) = \text{rk}_u^1(x) = \text{rk}_{I[u]}(t^x)$

- (b) let $\text{rk}_u^2 : X_u \rightarrow \{-1\} \cup \text{Ord} \cup \{\infty\}$
- (α) if $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \subseteq \{1\}$ (e.g., $n(x) = 0$) then let $\text{rk}_u^2(x) = \text{rk}_u^{2,\mathfrak{s}}(x) = \text{rk}_{I[u]}(t(x))$
- (β) if $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \not\subseteq \{1\}$ then let $\text{rk}_u^{2,\mathfrak{s}}(x) = -1$ (yes, -1).

7) We say that \mathfrak{s} is nice when every $I_u^\mathfrak{s}$ is non-trivial and $\pi_{u,w}$ is a function from I_v into I_u , i.e., the domain of $\pi_{u,v}^\mathfrak{s}$ is I_v .

8) $X_u^{<\alpha} := \{x \in X_u^\mathfrak{s} : \text{rk}_u^2(x) < \alpha\}$ and $X_u^{\leq\alpha} := \{x \in X_u^\mathfrak{s} : \text{rk}_u^2(x) \leq \alpha\}$. Note that $X_u^{\leq\alpha} = X_u^{<\alpha+1}$ when $\alpha < \infty$. Of course, we may write $X_u^{<\alpha,\mathfrak{s}}, X_u^{\leq\alpha,\mathfrak{s}}$ and note that $X_u^{<0} = \{x \in X_u^\mathfrak{s} : 0 \in \text{Rang}(\eta^x)\}$.

1.3 Definition. Assume \mathfrak{s} is a κ -p.o.w.i.s. and $u \in J^\mathfrak{s}$.

1) Let $G_u = G_u^\mathfrak{s} = G_u[\mathfrak{s}]$ be the group generated by $\{g_x : x \in X_u^\mathfrak{s}\}$ freely except the equations in $\Gamma_u = \Gamma_u^\mathfrak{s}$ where Γ_u consists of

- (a) $g_x^{-1} = g_x$, that is g_x has order 2, for each $x \in X_u$
- (b) $g_{y_1}g_{y_2} = g_{y_2}g_{y_1}$ when $y_1, y_2 \in X_u$ and $n(y_1) = n(y_2)$
- (c) $g_xg_{y_1}g_x^{-1} = g_{y_2}$ when $\otimes_{x,y_1,y_2}^{u,\mathfrak{s}}$, see below.

1A) Let $\otimes_{x,y} = \otimes_{x,y}^u = \otimes_{x,y}^{u,\mathfrak{s}}$ means that \otimes_{x,y_1,y_2} for some y_1, y_2 such that $y \in \{y_1, y_2\}$, see below.

1B) Let $\otimes_{x,y_1,y_2} = \otimes_{x,y_1,y_2}^u = \otimes_{x,y_1,y_2}^{u,\mathfrak{s}}$ means that:

- (a) $x, y_1, y_2 \in X_u$
- (b) $n(x) < n(y_1) = n(y_2)$
- (c) $y_1 \upharpoonright n(x) = y_2 \upharpoonright n(x)$
- (d) $\bar{t}^{y_1} = \bar{t}^{y_2}$
- (e) for $\ell < n(y_1)$ we have: $\eta^{y_1}(\ell) \neq \eta^{y_2}(\ell)$ iff $\ell = n(x) \wedge x = y_1 \upharpoonright n(x)$.

2) Let $G_u^{<\alpha} = G_u^{<\alpha,\mathfrak{s}}$ be defined similarly to $G_u^\mathfrak{s}$ except that it is generated only by $\{g_x : x \in X_u^{<\alpha}\}$, freely except the equations from $\Gamma_u^{<\alpha} = \Gamma_u^{<\alpha,\mathfrak{s}}$, where $\Gamma_u^{<\alpha}$ is the set of equations from Γ_u among $\{g_x : x \in X_u^{<\alpha}\}$.

Similarly $G_u^{\leq\alpha}, \Gamma_u^{\leq\alpha}$; note that $G_u^{\leq\alpha} = G_u^{<\alpha+1}, \Gamma_u^{\leq\alpha} = \Gamma_u^{<\alpha+1}$ if $\alpha < \infty$.

3) For $X \subseteq X_u$ let $G_{u,X} = G_{u,X}^\mathfrak{s}$ be the group generated by $\{g_y : y \in X\}$ freely except the equations in $\Gamma_{u,X} = \Gamma_{u,X}^\mathfrak{s}$ which is the set of equations from Γ_u mentioning only generators among $\{g_y : y \in X\}$.

- 1.4 *Observation.* 1) The sequence $\langle X_u^{<\alpha} : \alpha \leq \text{rk}(I_u^{\mathfrak{s}}) \rangle$ is \subseteq -increasing continuous.
 2) If $\ell \in \{1, 2\}$ and $x, y \in X_u$ are such that $x \neq y = x \upharpoonright n$ and $\ell \in \{1, 2\}$ then $\text{rk}_{\mathbf{p}}^\ell(y) \geq \text{rk}_{\mathbf{p}}^\ell(x)$ and if equality holds then $\text{rk}_u^1(x) = \infty = \text{rk}_u^1(y)$ or both are -1 and $\ell = 2$.
 3) If a partial order I is explicitly non-trivial then I is non-trivial.

Proof. Check.

1.5 *Observation.* For a κ -p.o.w.i.s. \mathfrak{s} .

- 1) $\otimes_{x,y}^{u,\mathfrak{s}}$ holds iff:

- (α) $x, y \in X_u$ and
- (β) $n(y) \geq n(x) + 1$.

- 2) If $x \in X_u^{\mathfrak{s}}$ then $\{(y_1, y_2) : \otimes_{x,y_1,y_2}^{u,\mathfrak{s}} \text{ holds}\}$ is a permutation of order two of $Y_{>n(x)} =: \{y \in X_n^{\mathfrak{s}} : n(y) > n(x)\}$.
 3) Moreover, the permutation in part (2) maps each $Y_{n+1} \setminus Y_n$ onto itself when $n \in [n(x), \omega)$ and so it maps $\Gamma_{Y_{>n}}$ onto itself when $n(*) \leq n < \omega$.
 4) If $\otimes_{x,y_1,y_2}^{u,\mathfrak{s}}$ then $y_1 \upharpoonright n(x) = y_2 \upharpoonright n(x)$ and $n(x) < n(y_1) = n(y_2)$.
 5) $\otimes_{x,y_1,y_2}^{u,\mathfrak{s}}$ iff $\otimes_{x,y_2,y_1}^{u,\mathfrak{s}}$.
 6) For $x, y \in X_u^{\mathfrak{s}}$, in the group $G_u^{\mathfrak{s}}$ the elements g_x, g_y commute except when $x \neq y \wedge (x = y \upharpoonright n(x) \vee y = x \upharpoonright n(y))$. In this case, if $n(x) < n(y)$ there is $y' \neq y$ such that

$$\otimes_{x,y,y'} \text{ so } n(y') = n(y) \text{ and } \eta^y(\ell) = \eta^{y'}(\ell) \Leftrightarrow \ell \in n(x).$$

Proof. Straight (details on (2),(3) see the proof of 1.9). $\square_{1.5}$

We first sort out how elements in $G_u^{\mathfrak{s}}$ and various subgroups can be (uniquely) represented as products of the generators.

1.6 Claim. Assume that \mathfrak{s} is a κ -p.o.w.i.s., $u \in J^*$ and $<^*$ is any linear order of X_u such that

$$\square \text{ if } x \in X_u, y \in X_u \text{ and } n(x) > n(y) \text{ then } x <^* y.$$

- 1) Any member of G_u is equal to a product of the form $g_{x_1} \dots g_{x_m}$ where $x_\ell <^* x_{\ell+1}$ for $\ell = 1, \dots, m-1$. Moreover, this representation is unique.
 2) Similarly for $G_u^{<\alpha}, G_u^{<\alpha}$ (using $X_u^{<\alpha}, X_u^{<\alpha}$ respectively instead X_u) hence $G_u^{<\alpha}, G_u^{<\alpha}$ are subgroups of G_u .

- 3) In part (1) we can replace G_u and X_u by $G = G_{u,X}$ and X respectively when $X \subseteq X_u$ is such that $[\{x, y_1, y_2\} \subseteq X_u \wedge \otimes_{x, y_1, y_2}^{u, s} \wedge \{x, y_1\} \subseteq X \Rightarrow y_2 \in X]$. Hence $G_{u,X}$ is equal to $\langle \{g_x : x \in X\} \rangle_{G_u}$.
- 4) If $g = g_{y_1} \dots g_{y_m}$ where $y_1, \dots, y_m \in X_u$ and $g = g_{x_1} \dots g_{x_n} \in G_u$ and $x_1 <^* \dots <^* x_n$ then $n \leq m$.
- 5) $\langle G_u^{<\alpha} : \alpha \leq \text{rk}(I_u^s), \alpha \text{ an ordinal} \rangle$ is an increasing continuous sequence of groups with last element $G_u^{<\infty}$.
- 6) $\{gG_u^{<0} : g \in G_u\}$ is a partition of G_u (to left cosets of G_u over $G_u^{<0}$).
- 7) If $<^1, <^2$ are two linear orders of X_u as in \square above and $G_u \models "g_{x_1} \dots g_{x_k} = g_{y_1} \dots g_{y_m}"$ and $x_1 <^1 \dots <^1 x_k$ and $y_1 <^2 \dots <^2 y_m$ (or just $x_1 <^1 \dots \hat{x}_k, n(y_1) \geq n(y_2) \geq \dots n(y_n)$ and $\langle y_\ell : \ell = 1, m \rangle$ is with no repetitions), then:

(α) $k = m$

(β) for every i we have $\{\ell : n(x_\ell) = i\} = \{\ell : n(y_\ell) = i\}$ and this set is a convex subset of $\{1, \dots, m\}$.

(So the only difference is permuting $g_{x_{\ell(2)}}, g_{x_{\ell(1)}}$ when $n(x_{\ell(1)}) = n(x_{\ell(2)})$).

- 8) If $I \subseteq I_u$ and $X = X_I$ then $G_{u,X} \cap G_u^{<0}$ is the subgroup of $G_{u,X}$ generated by $\{g_x : x \in X, \text{Rang}(\eta^x) \not\subseteq 1\}$, i.e., the (naturally defined) $G_I^{<0}$.
- 9) If $I_\ell \subseteq I_u^s$ for $\ell = 1, 2, 3$ (so $\leq_{I_\ell} = \leq_I \upharpoonright I_\ell$) and $I_1 \cap I_2 = I_3$ then $G_{I_1} \cap G_{I_2} = G_{I_3}$ and $G_{I_1}^{<0} \cap G_{I_2}^{<0} = G_{I_3}^{<0}$.

Proof. 1),2),3) Recall that each generator has order two. We can use standard combinatorial group theory (the rewriting process but below we do not assume knowledge of it); the point is that in the rewriting the number of generators in the word do not increase (so no need of $<^*$ being a well ordering).

We now give a full self-contained proof, for part of (2) we consider $G = G_u^{<\alpha}, X = X_u^{<\alpha} \subseteq X_u, \Gamma = \Gamma_u^{<\alpha}$ for α an ordinal or infinity and for part (1) and the rest of part (2) consider $G = G_u^{\leq\beta}, X = X_u^{\leq\beta} \subseteq X_u, \Gamma = \Gamma_u^{\leq\beta}$ for β an ordinal or infinity (recall that G_u, X_u is the case $\beta = \infty$). Now in parts (1),(2) for the set X , the condition from part (3) holds by 1.4(2).

[Why? So assume \otimes_{x, y_1, y_2}^u and e.g. $x, y_1 \in X_u^{<\alpha}$ and we should prove that $y_2 \in X_u^{<\alpha}$. If $y_1 = y_2$ this is trivial so assume $y_1 \neq y_2$, hence necessarily $y_1 \upharpoonright n(x) = x = y_2 \upharpoonright n(x)$ and $n(x) < n(y_1) = n(y_2)$ and $\bar{t}^{y_1} = \bar{t}^{y_2}$ and $\eta^{y_1}(\ell) = \eta^{y_2}(\ell) \Leftrightarrow \ell \neq n(x)$. If η^x is not constantly one then also η^{y_1} is not constantly one hence $y_2 \in X_u^{<0}$ so fine. If η^x is constantly one then $\alpha \geq \text{rk}_u^1(t^x) > \text{rk}_u^1(t^{y_1}) = \text{rk}_u^1(t^{y_2}) \geq \text{rk}_u^2(t^{y_2})$ hence $y_2 \in X_u^{\leq\alpha}$ so fine.]

So it is enough to prove part (3). Now recall that $G = G_{u,X}$ and

\otimes_1 every member of G can be written as a product $g_{x_1} \dots g_{x_n}$ for some $n <$

$\omega, x_\ell \in X$

[Why? As the set $\{g_x : x \in X\}$ generates G .]

- ⊗₂ if in $g = g_{x_1} \dots g_{x_n}$ we have $x_\ell = x_{\ell+1}$ then we can omit both
[Why? As $g_x g_x = e_G$ for every $x \in X$ by clause (a) of Definition 1.3(1)]
- ⊗₃ if $1 \leq \ell < n$ and $g = g_{x_1} \dots g_{x_n}$ and we have $x_{\ell+1} <^* x_\ell$ and $m \in \{1, \dots, n\} \setminus \{\ell, \ell+1\} \Rightarrow y_m = x_m$ then we can find $y_\ell, y_{\ell+1} \in X$ such that $g = g_{y_1} \dots g_{y_n}$ and $y_\ell <^* y_{\ell+1}$ and, in fact, $y_{\ell+1} = x_\ell$.

[Why does ⊗₃ hold? By Definition 1.3(1) and Observation 1.5(6) one of the following cases occurs.

Case 1: $g_{x_\ell}, g_{x_{\ell+1}}$ commutes.

Let $y_\ell = x_{\ell+1}, y_{\ell+1} = x_\ell$.

Case 2: Not Case 1 but $\otimes_{x_{\ell+1}, x_\ell}^{u, 5}$, see Definition 1.3(1A).

By clause (b) of Definition 1.3(1) we have $n(x_{\ell+1}) < n(x_\ell)$. So by \square of the assumption of the present claim we have $x_\ell <^* x_{\ell+1}$, contradiction.

Case 3: Not case 1 but $\otimes_{x_\ell, x_{\ell+1}}^{u, 5}$, see Definition 1.3(1B).

By 1.5(6) there is $y_\ell \in X$ such that $n(y_\ell) = n(x_{\ell+1}) > n(x_\ell)$, $\bar{t}^{y_\ell} = \bar{t}^{x_{\ell+1}}$ and $i < n(x_{\ell+1}) \Rightarrow (\eta^{y_\ell}(i) = \eta^{x_{\ell+1}}(i)) \equiv (i \neq n(x_\ell))$.

Let $y_{\ell+1} = x_\ell$, clearly $y_{\ell+1}, y_\ell \in X$. By Definition 1.3(1), we have $g_{x_\ell} g_{x_{\ell+1}} g_{x_\ell}^{-1} = g_{y_\ell}$ hence $g_{x_\ell} g_{x_{\ell+1}} = g_{y_\ell} g_{x_\ell} = g_{y_\ell} g_{y_{\ell+1}}$ and clearly $n(y_{\ell+1}) = n(x_\ell) < n(y_\ell)$ hence $y_\ell <^* x_\ell = y_{\ell+1}$, so we are done.

The three cases exhaust all possibilities hence ⊗₃ is proved.]

⊗₄ every $g \in G$ can be represented as $g_{x_1} \dots g_{x_n}$ with $x_1 <^* x_2 <^* \dots <^* x_n$.

[Why? Without loss of generality g is not the unit of G . By ⊗₁ we can find $x_1, \dots, x_n \in X_1$ such that $g = g_{x_1} \dots g_{x_n}$ and $n \geq 1$. Choose such a representation satisfying

- ⊗ (a) with minimal n and
- (b) for this n , with minimal $m \in \{1, \dots, n+1\}$ such that $x_m <^* \dots <^* x_n$
and $1 \leq \ell < m \leq n \Rightarrow \bigwedge_{\ell=1}^{m-1} x_\ell <^* x_m$, and
- (c) for this pair (n, m) if $m > 2$ then with maximal ℓ where $\ell \in \{1, \dots, m-1\}$ satisfies x_ℓ is $<^*$ -maximal among $\{x_1, \dots, x_{m-1}\}$ that is $k \in \{1, \dots, m-1\} \Rightarrow x_k \leq^* x_\ell$.

Easily there is such a sequence (x_1, \dots, x_n) , noting that $m = n + 1$ is O.K. for (b) and there is ℓ as in $\otimes(c)$.

By \otimes_2 and clause (a) of \otimes we have $x_\ell \neq x_{\ell+1}$ when ℓ from $\otimes(c)$ is well defined, i.e., if $m > 2$).

Now $m = 2$ is impossible (as then $m = 1$ can serve), if $m = 1$ we are done, and if $m > 2$ then ℓ is well defined and $\ell = m - 1$ is impossible (as then $m - 1$ can serve instead m). Lastly by \otimes_3 applied to this ℓ , we could have improved ℓ to $\ell + 1$, contradiction.]

\otimes_5 the representation in \otimes_4 is unique.

[Why does \otimes_5 hold? Assume toward contradiction that $g_{x'_1} \dots g_{x'_{n_1}} = g_{y'_1} \dots g_{y'_{n_2}}$ where $x'_1 <^* \dots <^* x'_{n_1}$ and $y'_1 <^* \dots <^* y'_{n_2}$ and $(x'_1, \dots, x'_{n_1}) \neq (y'_1, \dots, y'_{n_2})$. Without loss of generality among all such examples, $(n_1 + n_2 + 1)^2 + n_1$ is minimal.

Recall $Y_n =: \{x \in X : n(x) = n\}$.

So $\langle Y_n : n < \omega \rangle$ is a partition of X .

For $k \leq m < \omega$ let $X^{<k,m>} = \bigcup \{Y_\ell : k \leq \ell < m\}$ and let $G^{<k,m>}$ be the group generated by $\{g_x : x \in X^{<k,m>}\}$ freely except the equations in $\Gamma^{<k,n>}$, i.e., the equations from $\Gamma_{u,X^{<k,m>}}$, i.e., the equations from Definition 1.3(4) mentioning only its generators, $\{y_x : x \in X^{<k,m>}\}$. Now clearly if $\otimes_{x,y_1,y_2}^{u,s}$, see Definition 1.3(1B) then $n < \omega \Rightarrow [y_2 \in Y_n \equiv y_2 \in Y_n]$ so the set $X^{<k,m>} \subseteq X$ satisfies the requirement in part (3) of 1.6 which we are proving; so what we have proved for X holds for $X^{<k,m>}$. In particular $\otimes_1 - \otimes_4$ above gives that for every $g \in G^{<k,m>}$ there are n and $x_1 <^* \dots <^* x_n$ from $X^{<k,m>}$ such that $G^{<n,m>} \models "g = g_{x_1} \dots g_{x_n}"$. Also it is enough to prove the uniqueness for $G^{<k,m>}$ (for every $k \leq m < \omega$), i.e., we can assume $x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2} \in X^{<k,m>}$ as if the equality holds (though $\langle x'_1, \dots, x'_{n_1} \rangle \neq \langle y'_1, \dots, y'_{n_2} \rangle$), finitely many equations of $\Gamma_{u,X}$ implies the undesirable equation and for some $k \leq m < \omega$ they are all from $\Gamma^{<k,m>}$ and $\{x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2}\} \subseteq X$, hence already in $G^{<k,m>}$ we get this undesirable equation.

Now for $k < m < \omega$ and $x \in Y_k$ let $\pi_x^{k,m}$ be the following permutation of $X^{<k+1,m>}$: it maps $y_1 \in X^{<k+1,m>}$ to y_2 if $\otimes_{x,y_1,y_2}^{u,s}$.

It is easy to check that

\square_1 For k, m, x as above,

- (i) $\pi_x^{k,m}$ is a permutation of $X^{<k+1,m>}$ which maps $\Gamma^{<k+1,m>}$ onto itself
- (ii) $\pi_x^{k,m}$ induce an automorphism $\hat{\pi}_x^{k,m}$ of $G^{<k,m>}$: the one mapping g_{y_1} to g_{y_2} when $\pi_x^{k,m}(y_1) = y_2$
- (iii) the automorphisms $\hat{\pi}_x^{k,m}$ of $G^{<k,m>}$ for $x \in Y_k$ pairwise commute
- (iv) the automorphism $\hat{\pi}_x^{k,m}$ of $G^{<k,m>}$ is of order two.

We prove this revised formulation of the uniqueness by induction on $m - k$.
Note that

(*) if $x \in Y_k, y \in Y_\ell$ and $x <^* y$ then $\ell \leq k$.

If $m - k = 0$, then $G^{< k, m >}$ is the trivial group so the uniqueness is trivial.

Also the case $k = m - 1$ is trivial too as in this case $G^{< k, m >}$ is actually a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{g_x : x \in Y_k\}$, well in additive notation so the uniqueness is clear.

So assume that $m - k \geq 2$, now we need

$\square_{k,m}^2$ if $x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2}$ from $X^{< k, m >}$ are as above in $G^{< k, m >}$ then $\langle x'_1, \dots, x'_{n_1} \rangle = \langle y'_1, \dots, y'_{n_2} \rangle$.

We can prove the induction step by 1.7 below.

So 1), 2), 3) holds.

4) Included in the proof of \otimes_4 inside the proof of parts (1), (2), (3).

5) For $\alpha < \beta \leq \infty$, clearly $X_u^{< \alpha} \subseteq X_u^{< \beta}$ and $\Gamma_u^{< \alpha} \subseteq \Gamma_u^{< \beta}$ hence there is a homomorphism from $G_u^{< \alpha}$ into $G_u^{< \beta}$. This homomorphism is one-to-one (because of the uniqueness clause in part (2)) hence the homomorphism is the identity. So the sequence is \subseteq -increasing, the \subset follows by part (1), the uniqueness we have $\text{rk}_I(t) = \alpha < \infty \Rightarrow g_{(\langle t \rangle, < \rangle)} \in G_u^{< \alpha+1} \setminus G_u^{< \alpha}$.

6), 7), 8), 9) Easy.

$\square_{1.6}$

1.7 Observation. Assume that

- (a) G is a group
- (b) f_t is an automorphism of G for $t \in J$
- (c) $f_t, f_s \in \text{Aut}(G)$ commute for any $s, t \in J$.

Then there are K and $\langle g_t : t \in J \rangle$ such that

- (α) K is a group
- (β) G is a normal subgroup of K
- (γ) H is generated by $G \cup \{g_t : t \in J\}$
- (δ) if $a \in G$ and $t \in G$ then $g_t a g_t^{-1} = f_t(a)$
- (ε) if $<_*$ is a linear order of J then every member of K has a one and only one representation as $x g_{t_1}^{b_1} g_{t_2}^{b_2} \dots g_{t_n}^{b_n}$ where $x \in G, n < \omega, t_1 <_* \dots <_* t_n$ are from J and $b_1, \dots, b_n \in \mathbb{Z} \setminus \{0\}$.

Proof. A case of semi-direct product, see below. (It is also a case of repeated HNN extensions). $\square_{1.7}$

1.8 Definition/Claim. 1) Assume G_1, G_2 are groups and π is a homomorphism from G_1 into $\text{Aut}(G_2)$, we define the sem-direct product $G = G_1 *_\pi G_2$ as follows:

- (a) the set of elements is $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$
- (b) the product operation is $(g_1, g_2) * (h_1, h_2) = (g_1 h_1, g_2^{\pi(h_1)} h_2)$ where
 - (α) $g_2^{\pi(h_1)}$ is the image of g_2 by the automorphism $\pi(h_1)$ of G_2
 - (β) $g_1 h_1$ is a G_1 -product
 - (γ) $g_2^{\pi(h_1)} h_2$ is a G_2 -product.

2)

- (a) such group G exists
- (b) in G every member has one and only one representation as $g'_1 g'_2$ where $g'_1 \in G_1 \times \{e_{G_2}\}, g'_2 \in \{e_{G_1}\} \times G_2$
- (c) the mapping $g_1 \mapsto (g_1, e)$ embeds G_1 into G
- (d) the mapping $g_2 \mapsto (e, g_2)$ embeds G_2 into G
- (e) so up to renaming, for each $h_1 \in G_1$ conjugating by it (i.e. $g \mapsto h_1^{-1} g h_1$) inside G acts on G_2 as the automorphism $\pi(h_1)$ of G_1 .

3) If H_1, H_2 is a subgroup of G_1, G_2 respectively, and $g_1 \in H_1 \Rightarrow \pi(g_1)$ maps H_2 onto itself and $\pi' : H_1 \rightarrow \text{Aut}(H_2)$ is $\pi'(x) = \pi(x) \upharpoonright H_2$ then $\{(h_1, h_2) : h_1 \in H_1, h_2 \in H_2\}$ is a subgroup of $G_1 *_\pi G_2$ and is in fact $H_1 *_\pi H_2$; we denote π' by $\pi_{[H_1/H_2]}$.

4) If the pairs (H_1^a, H_2^a) and (H_1^b, H_2^b) are as in part (3) and $H_1^c := H_1^a \cap H_1^b, H_2^c := H_2^a \cap H_2^b$ then the pair (H_1^c, H_2^c) is as in part (3) and $(H_1^a *_\pi [H_1^a, H_2^a] H_2^a) \cap (H_1^b *_\pi [H_1^b, H_2^b] H_2^b) = (H_1^c *_\pi [H_1^c, H_2^c] H_2^c)$.

Proof. Known and straight. $\square_{1.8}$

1.9 Claim. Let \mathfrak{s} be a κ -p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $I_u = I_u^{\mathfrak{s}}$ be non-trivial.

- 1) If $0 \leq \alpha < \infty$ then the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\alpha+1}$.
- 2) If $\alpha = \text{rk}(I_u)$ then the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\infty} = G_u^{<\alpha}$.

Proof. 1) First

(*)₁ if $x \in X_u$ and $\text{rk}_u^2(x) = \alpha$ then conjugation by g_x in G_u maps $\{g_y : y \in X_u^{<\alpha}\} = \{g_y : y \in X_u \text{ and } \text{rk}_u^2(y) < \alpha\}$ onto itself.

[Why? As $g_x = g_x^{-1}$ it is enough to prove for every $y \in X_u^{<\alpha}$ that: $g_x g_y g_x^{-1} \in X_u^{<\alpha}$. Now for each such y , one of the following cases occurs.

Case (i): g_x, g_y commutes so $g_x g_y g_x^{-1} = g_y \in X_u^{<\alpha}$.

In this case the desired conclusion holds trivially.

Case (ii): $n(y) \leq n(x)$ and not case (i).

As case (i) does not occur, necessarily $n(y) < n(x)$ and $y = x \upharpoonright n(y)$ by 1.5(6). Also it follows that $t_{n(x)}^x <_{I_u[s]} t_{n(y)}^y$, i.e., $t(x) <_{I_u[s]} t(y)$ but $\text{rk}_u^2(x) = \alpha$ hence $\text{rk}_u^2(y) \in \{-1\} \cup [\alpha + 1, \infty]$. However we are assuming $y \in X_u^{<\alpha}$ hence necessarily $y \in X_u^{<0}$, so $\langle \eta^y(\ell) : \ell < n(y) \rangle$ is not constantly 1 hence $\langle \eta^x(\ell) : \ell < n(x) \rangle$ is not constantly 1 hence $\text{rk}_u^2(x) = 0$, contradiction.

Case (iii): $n(y) > n(x)$ and not case (i).

As in case (ii) by 1.5(6) we have $x = y \upharpoonright n(x)$.

Clearly $t(y) = t_{n(y)}^y <_{I_u[s]} t_{n(x)}^y = t_{n(x)}^x = t(x)$ so as $\text{rk}_u^2(x) \geq 0$ necessarily $\text{rk}_u^1(x) = \text{rk}_u^2(t(x)) = \alpha \in [0, \infty)$ hence $\text{rk}_{I_u}(t^y) < \text{rk}_{I_u}(t^x) = \alpha$ and so $\text{rk}_u^2(y) \leq \text{rk}_u^1(t^y) < \alpha$.

Let $y_1 = y$ and by 1.5(1),(6) and Definition 1.3(1A) there is y_2 such that $\otimes_{x, y_1, y_2}^{u, 5}$ hence $G_u \models g_x g_y g_x^{-1} = g_{y_2}$ and $\bar{t}^y = \bar{t}^{y_1} = \bar{t}^{y_2}$, so $\text{rk}_u^2(y_2) \leq \text{rk}_u^1(y_2) = \text{rk}_u^1(t^{y_2}) = \text{rk}_u^1(t^{y_1}) < \alpha$ hence $y_2 \in X_u^{<\alpha}$ and so $g_{y_2} \in G_u^{<\alpha}$ so we are done.

So (*)₁ holds.]

Now by (*)₁ it follows that g_x normalize $G_u^{<\alpha}$ for every member g_x of $\{g_x : \text{rk}_u^2(x) = \alpha\}$, hence clearly $\text{nor}_{G_u}(G_u^{<\alpha}) \supseteq (G_u^{<\alpha}) \cup \{g_x : \text{rk}_u^2(x) = \alpha \text{ and } x \in X_u\}$ but the latter generates $G_u^{<\alpha+1}$ hence

$$(*)_2 \text{ nor}_{G_u}(G_u^{<\alpha}) \supseteq G_u^{<\alpha+1}.$$

Second assume $g \in G_u \setminus G_u^{<\alpha+1}$, let $<^*$ be a linear ordering of X_u as in \square of 1.6; so we can find $k < \omega$ and $x_1 <^* \dots <^* x_k$ from X_u such that $g = g_{x_1} g_{x_2} \dots g_{x_k}$ and so it suffices to prove by induction on k that if $g = g_{x_1} \dots g_{x_k} \in G_u \setminus G_u^{<\alpha+1}$ then $g \notin \text{nor}_{G_u}(G_u^{<\alpha})$. By 1.6(2),(4) without loss of generality $x_1 <^* \dots <^* x_k$. As $g \notin G_u^{<\alpha+1}$ necessarily not all the x_m 's are from $X_u^{<\alpha+1}$ hence for some m , $g_{x_m} \notin G_u^{<\alpha+1}$.

(*)₃ without loss of generality $x_1, x_k \notin G_u^{<\alpha+1}$.

[Why? So assume $x_k \in G_u^{<\alpha+1}$ hence

(a) $x_k \in \text{nor}_{G_u}(G_u^{<\alpha})$ (as we have already proved $G_u^{<(\alpha+1)} \subseteq \text{nor}_{G_u}(G_u^{<\alpha})$)

(b) $\text{nor}_{G_u}(G_u^{<\alpha})$ is a subgroup of G_u hence

(c) $g = g_{x_1} \dots g_{x_{k-1}} g_{x_k} \in \text{nor}_{G_u}(G_u^{<\alpha})$ iff $g_{x_1} \dots g_{x_{k-1}} \in \text{nor}_{G_u}(G_u^{<\alpha})$.

By the induction hypothesis on k we get are done. Similarly if $g_{x_1} \in G_u^{<\alpha+1}$ then derive $g \in \text{nor}_{G_u}(G_u^{<\alpha})$ iff $g_{x_2} \dots g_{x_k} \in \text{nor}_{G_u}(G_u^{<\alpha})$ to finish.]

Now we can find $t^* \in I_u$ such that

- (*)₄ (a) $t^* <_{I_u} t(x_1)$
- (b) $\text{rk}_{I_u}(t^*) \geq \alpha$
- (c) $t^* \notin \{t_\ell(x) : x \in \{x_1, \dots, x_k\} \text{ and } \ell \in \{0, \dots, n(x)\}\}$.

[Why? As we assume that \mathfrak{s} is nice which implies that each I_u is non-trivial, see Definition 1.1(6) and Definition 1.2(7).]

Let $m(*)$ be maximal such that $1 \leq m(*) \leq k$ and $(\exists i)(x_{m(*)} = x_1 \upharpoonright i)$.

Now we choose $y \in X_u^{\mathfrak{s}}$ as follows:

- (*)₅ (a) $\bar{t}^y = \bar{t}^{x_{m(*)}} \wedge \langle t^* \rangle$
- (b) $\eta^y \upharpoonright n(x_{m(*)}) = \eta^{x_{m(*)}}$
- (c) $\eta^y(n(x_{m(*)})) = 0$.

Note that

- (*)₆ $y \in X_u^{<0}$ and $n(y) = n(x_{m(*)}) + 1$ and
- (*)₇ $n(x_1) \geq \dots \geq n(x_{m(*)}) \geq n(x_{m(*)+1}) \geq \dots \geq n(x_k)$.

We now try to define $\langle y_\ell : \ell = 1, \dots, k+1 \rangle$ by induction on ℓ as follows :

- (*)₈ $y_1 = y$ and $G_u \models g_{x_\ell}^{-1} g_{y_\ell} g_{x_\ell} = g_{y_{\ell+1}}$ if well defined.

So

- (*)₉ $y_\ell = y$ for $\ell = 1, \dots, m(*)$ and is well defined.

[Why? We prove it by induction on ℓ . For $\ell = 1$ this is given. So assume that this holds for ℓ and we shall prove it for $\ell + 1$ when $\ell + 1 \leq m(*)$. Now $\neg(\bar{t}^y \triangleleft \bar{t}^{x_\ell})$ by the choice of t^* (and y) and hence $\neg(y = x_\ell \upharpoonright n(y) \wedge n(y) < n(x_\ell))$ and we also have $\neg(x_\ell = y \upharpoonright n(x_\ell) \wedge n(x_\ell) < n(y))$ as otherwise $x_\ell = x_{m(*)} \upharpoonright n(x_\ell)$ but $n(x_\ell) \geq n(x_{m(*)})$ as $x_\ell <^* x_{m(*)}$ hence $x_\ell = x_{m(*)}$, but $\ell \neq m$ hence $x_\ell \neq x_{m(*)}$, contradiction. Together by 1.5(6) the elements g_y, g_{x_ℓ} commute so as by the induction hypothesis $y_\ell = y$ it follows that $y_{\ell+1} = y$ so we are done.]

Now:

- (*)₁₀ $y_{m(*)+1}$ is well defined and satisfies (*)₅(a), (b) and (*)₅(c) when we replace 0 by 1.

[Why? By the definition of G_u in 1.3(1),(1B).]

- (*)₁₁ $y_{m(*)+1} \notin G_u^{<\alpha}$.
 [Why? By (*)₃, $x_1 \notin G_u^{<\alpha+1}$ hence η^{x_1} is constantly 1; but $x_{m(*)} = x_1 \upharpoonright n(x_{m(*)})$ hence $\eta^{x_{m(*)}}$ is constantly one. Now $\eta^{y_{m(*)+1}} = \eta^{x_{m(*)}} \wedge \langle 1 \rangle$ by (*)₁₀ hence $\eta^{y_{m(*)+1}}$ is constantly one. So $\text{rk}_u^2(y_{m(*)+1}) = \text{rk}_{I[u]}(t^{y_{m(*)+1}}) = \text{rk}_u(t^*) \geq \alpha$ so we are done.]
- (*)₁₂ if $\ell \in \{m(*) + 1, \dots, k + 1\}$ then $y_\ell = y_{m(*)+1}$ and y_ℓ is well defined.
 [Why? We prove this by induction on ℓ . For $\ell = m(*) + 1$ this is trivial by (*)₁₀. For $\ell + 1 \in \{m(*) + 2, \dots, k + 1\}$, it is enough to prove that $y_{m(*)+1}, x_\ell$ commute. Now $\neg(\bar{t}^{y_{m(*)+1}} \triangleleft \bar{t}^{x_\ell})$ because $\ell g(\bar{t}^{y_{m(*)+1}}) = \ell g(\bar{t}^y) = \ell g(\bar{t}^{x_{m(*)}}) + 1 \geq \ell g(\bar{t}^{x_\ell}) + 1 > \ell g(\bar{t}^{x_\ell})$ hence $\neg(y_{m(*)+1} = x_\ell \upharpoonright n(y_{m(*)+1}) \wedge n(y_{m(*)+1}) < n(x_\ell))$; also $\neg(x_\ell = y_{m(*)+1} \upharpoonright n(x_\ell) \wedge n(x_\ell) < n(y_{m(*)+1}))$ as otherwise this contradicts the choice of $m(*)$. So by 1.5(6) they commute indeed.]
- (*)₁₃ $g^{-1}g_y g = g_{y_{k+1}}$.
 [Why? We can prove by induction on $\ell = 1, \dots, k + 1$ that $(g_1 \dots g_{\ell-1})^{-1} g_y (g_1 \dots g_{\ell-1}) = g_{y_\ell}$, by the definition of the y_ℓ 's, i.e., by (*)₈ and they are well defined by (*)₉ + (*)₁₀ + (*)₁₂.]
- (*)₁₄ $g^{-1}g_y g = g_{m(*)+1}$.
 [Why? By (*)₁₂ and (*)₁₃.]
- (*)₁₅ $g^{-1}g_y g \notin G_u^{<\alpha}$.
 [Why? By (*)₁₄ + (*)₁₁.]

So by (*)₆ we have $g_y \in G_u^{<0} \subseteq G_u^{<\alpha}$ and by (*)₁₅ we have $g^{-1}g_y g \notin G_u^{<\alpha}$ hence g does not normalize $G_u^{<\alpha}$, so we have carried the induction on k . As g was any member of $G_u \setminus G_u^{<(\alpha+1)}$ we get $\text{nor}_{G_u}(G_u^{<\alpha}) \subseteq G_u^{<(\alpha+1)}$.

Together with (*)₂ we are done.

2) Follows. □_{1.9}

§2 EASIER GROUP

The $G_u^\mathfrak{s}$'s from §1 has long towers of normalizers but the “base”, $G_u^{<0,\mathfrak{s}}$ is in general of large cardinality. Hence we replace below $G_u^\mathfrak{s}$ by $K_u^\mathfrak{s}$ and $G_u^{<0,\mathfrak{s}}$ by $H_u^\mathfrak{s}$.

2.1 Definition. Let \mathfrak{s} be a κ -p.o.w.i.s.

1) For $u \in J^\mathfrak{s}$:

- (a) recall 1.6(6): $\mathcal{A}_u = \mathcal{A}_u^\mathfrak{s} = \{gG_u^{<0} : g \in G_u\}$ is a partition of G (to left cosets of $G_u^{<0}$ inside G_u);
- (b) we define for every $f \in G_u$ a permutation ∂_f of \mathcal{A}_u defined by $\partial_f(g_1G_u^{<0}) = (fg_1)G_u^{<0}$, we may write it also as $f(g_1G)$
- (c) let $L_u = L_u^\mathfrak{s}$ be the group generated by $\{h_\mathbf{a} : \mathbf{a} \in \mathcal{A}_u\}$ freely except $h_\mathbf{a}h_\mathbf{b} = h_\mathbf{b}h_\mathbf{a}$ and $h_\mathbf{a}^{-1} = h_\mathbf{a}$ for $\mathbf{a}, \mathbf{b} \in \mathcal{A}_u$; for $g \in G_u$ let $h_g = h_{gG_u^{<0}}$
- (d) let $\mathbf{h}_u = \mathbf{h}_u^\mathfrak{s}$ be the homomorphism from G_u into the automorphism group of L_u such that $f \in G_u \wedge \mathbf{a} \in \mathcal{A}_u \Rightarrow (\mathbf{h}_u(f))(h_\mathbf{a}) = h_{f\mathbf{a}}$
- (e) let $K_u = K_u^\mathfrak{s}$ be $G_u *_{\mathbf{h}_u} L_u$, the twisted product of G_u, L_u with respect to the homomorphism \mathbf{h}_u , see 1.8, and we identify G_u with $G_u \times \{e_{L_u}\}$ and L_u with $\{e_{G_u}\} \times L_u$
- (f) let $H_u = \{(e_{G_u}, h_{e_{G_u}^{<0}}), (e_{G_u}, e_{L_u})\}$, a subgroup of K_u and let $h_* := h_{e_{G_u}} = h_{e_{G_u}G_u^{<0}} \in L_u$, i.e. the pair $(e_{G_u}, g_{e_{G_u}^{<0}})$, this is the unique member of H_u which is not the unit.

2) For $\alpha \leq \infty$ let $K_u^{<\alpha} = K_u^{<\alpha,\mathfrak{s}}$ be the subgroup $\{(g, h) : g \in G_u^{<\alpha} \text{ and } h \in L_u\}$ of K_u . Similarly $K_u^{\leq\alpha} = K_u^{\leq\alpha,\mathfrak{s}}$.

3) For $u \in J^\mathfrak{s}$ let

- (a) $D_u = D_u^\mathfrak{s} = \{(v, g) : v \leq_{J[\mathfrak{s}]} u \text{ and } g \in K_v^\mathfrak{s}\}$
- (b) $Z_u^0 = \{(\bar{t}, \eta) : \bar{t} = \langle t_\ell : \ell \leq n \rangle, n < \omega, t_\ell \in I \text{ for each } \ell < n \text{ and } \eta \in {}^n 2\}$ and let $z = (\bar{t}^z, \eta^z) = (\langle t_\ell^z : \ell \leq n \rangle, \eta^z)$ and $n(z) = n$ for $z \in Z_u^0$; this is compatible with Definition 1.2(4); note that here \bar{t} is not necessarily decreasing
- (c) $Z_u^1 := \{\langle x_\ell : \ell < k \rangle : k < \omega, \text{ each } x_\ell \text{ is from } Z_u^0\}$ and let $z = (\langle x_\ell^z : \ell < k(z) \rangle)$ if $z \in Z_u^1$
- (d) $Z_u := Z_u^0 \cup Z_u^1$
- (e) for $z \in Z_u$ we define $\text{his}(z)$, a finite subset of I_u by
 - (α) if $z = (\langle t_\ell : \ell \leq n \rangle, \eta) \in Z_u^0$ then $\text{his}(z) = \{t_\ell : \ell \leq n\}$
 - (β) if $z \in Z_u^1$ say $z = (\langle \langle t_\ell^k : \ell \leq \ell_k \rangle, \eta^k \rangle : k < k^*) \in Z_u^1$ then $\text{his}(z) = \{t_\ell^k : k < k^* \text{ and } \ell \leq \ell_k\}$

- (f) for $z = Z_u$ let $n(z) = \Sigma\{\ell_k : k < k^*\}$ if $z = \langle (\langle t_\ell^k : \ell \leq \ell_k \rangle, \eta^k) : k < k^* \rangle \in Z_u^1$ and $n(z)$ is already defined if $z \in Z_u^0$ in clause (b).

2.2 Observation. In Definition 2.1.

- 1) For $u \in J^5$, K_u is well defined and G_u, L_u are subgroups of K_u (after the identification).
- 2) For $I \subseteq I_u^5$ let $L_{u,I}^5$ be the subgroup of L_u^5 be generated by $\{h_{gG_u^{<0}} : g \in G_{u,X_I}^5\}$. If $I_1, I_2 \subseteq I_u^5$ then $L_{u,I_1}^5 \cap L_{u,I_2}^5 = L_{u,I_1 \cap I_2}^5$.
- 3) For $I \subseteq I_u^5$ let $K_{u,I}^5$ be the subgroup of K_u^5 generated by $G_{u,X_I}^5 \cup L_{u,I}^5$. Then
 - (a) G_{u,X_I}^5 normalized $L_{u,I}^5$ inside K_u^5
 - (b) $K_{u,I}^5$ is $G_{u,X_I}^5 *_{\pi} L_{u,I}^5$ for the natural π .

Also

- (c) if $I_1, I_2 \subseteq I_u^5$ then $K_{u,I_1}^5 \cap K_{u,I_2}^5 = K_{u,I_1 \cap I_2}^5$.

Proof. Easy (recall 1.6(8),(9), 1.8(2),(3)).

- 2.3 Definition.** 1) If I is a partial order then ${}^k I$ is the set of $\bar{t} = \langle t_\ell : \ell < k \rangle$ where $t_\ell \in I$.
- 2) If $\bar{t} \in {}^k I$ then $\text{tp}_{\text{qf}}(\bar{t}, \emptyset, I) = \{(\iota, \ell_1, \ell_2) : \iota = 0 \text{ and } I \models t_{\ell_1} < t_{\ell_2} \text{ or } \iota = 1 \text{ and } t_{\ell_1} = t_{\ell_2} \text{ or } \iota = 2 \text{ and } I \models t_{\ell_1} > t_{\ell_2} \text{ and } \iota = 3 \text{ if none of the previous cases}\}$.
- 2A) Let $\mathcal{S}^k = \{\text{tp}_{\text{qf}}(\bar{t}, \emptyset, I) : \bar{t} \in {}^k I \text{ and } I \text{ is a partial order}\}$.
- 3) We say $\bar{t} \in {}^k I$ realizes $p \in \mathcal{S}^k$ when $p = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, I)$.
- 4) If $k_1 < k_2$ and $p_2 \in \mathcal{S}^{k_2}$ then $p_1 := p_2 \upharpoonright k_1$ is the unique $p_1 \in \mathcal{S}^{k_1}$ such that if $p_2 = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, I)$ then $p_1 = \text{tp}_{\text{qf}}(\bar{t} \upharpoonright k_1, \emptyset, I)$.

Remark. Below each member of $\Lambda_k^0, \Lambda_k^1, \Lambda_k^2$ will be a description of an element of $G_u^5, \mathcal{A}_u^5, K_u^5$ respectively from a k -tuple of members of I_u^5 . Of course, a member of Z_u^5 is a description of a generator of K_u^5 .

- 2.4 Definition.** 1) For $k < \omega$ let $\Lambda_k^0 = \cup\{\Lambda_{k,p}^0 : p \in \mathcal{S}^k\}$ where for $p \in \mathcal{S}^k$ we let $\Lambda_{k,p}^0$ be the set of sequences of the form $\langle (\bar{\ell}_j, \eta_j) : j < j(*) \rangle$ such that:

- (a) for each j for some $n = n(\bar{\ell}_j, \eta_j)$ we have $\bar{\ell}_j = \langle \ell_{j,i} : i \leq n(\bar{\ell}_j, \eta_j) \rangle$ is a sequence of numbers $< k$ of length $n + 1$ such that $p = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, I) \Rightarrow \langle t_{\ell_{j,i}} : i \leq n(\bar{\ell}_j, \eta_j) \rangle$ is decreasing
- (b) for each $j, \eta_j \in {}^n 2$ where $n = n(\bar{\ell}_j, \eta_j)$.

2) For any p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}, \bar{t} \in {}^k(I_u)$ and $\rho = \langle (\bar{\ell}_j, \eta_j) : j < j(*) \rangle \in \Lambda_k^0$, let $g_{\bar{t}, \rho}^u = g_{\bar{t}, \rho}^{u, \mathfrak{s}} = (\dots g_{(\bar{t}j, \eta_j)} \dots)_{j < j(*)}$, the product in $G_u \subseteq K_u$ (so if $j(*) = 0$ it is $e_{G_u} = e_{K_u}$) where

- (a) $\bar{t}^j = \text{seq}_{\rho, j}(\bar{t}) := \langle t_{\ell_j, i} : i \leq n(\ell_j, \eta_j) \rangle$
- (b) if \bar{t}^j is decreasing (in I_u) then $g_{(\bar{t}j, \eta_j)} \in G_u \subseteq K_u$ is already well defined, if not then $g_{(\bar{t}j, \eta_j)} = e_{K_u}$.

2A) For a p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}, t \in {}^k(I_u^{\mathfrak{s}})$ and $\rho = \langle (\bar{\ell}_j, \eta_j) : j < j(*) \rangle \in \Lambda_k^0$ let $z_{\bar{t}, \rho}^u = z_{\bar{t}, \rho}^{u, \mathfrak{s}}$ be the following member of $Z_u^{1, \mathfrak{s}}$: it is $\langle x_{\bar{t}, \rho, j} : j < j(*) \rangle$ where $x_{\bar{t}, \rho, j} = x_{\bar{t}, (\bar{\ell}_j, \eta_j)} = (\langle t_{\ell_j, i} : i \leq n(\bar{\ell}_j, \eta_j) \rangle, \eta_j)$. For $p \in \mathcal{S}^k$ and $\rho = \langle (\bar{\ell}_j, \eta_j) : j < j(*) \rangle \in \Lambda_{k, p}^0$ let $\text{supp}(\rho) = \cup \{ \text{Rang}(\bar{\ell}_j) : j < j(*) \}$ and if $\bar{t} \in {}^k(I_u^{\mathfrak{s}})$ let $\text{sup}(\bar{t}, \rho) = \{ t_\ell : \ell \in \text{supp}(\rho) \}$.

2C) We say $\rho \in \Lambda_k^0$ is p -reduced when: $p \in \mathcal{S}^k$ and for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $t \in {}^k(I_u^{\mathfrak{s}})$ realizing p (in $I_u^{\mathfrak{s}}$), for no $\rho' \in \Lambda_k^0$ do we have $\text{supp}(\rho') \subset \text{supp}(\rho)$ and $g_{\bar{t}, \rho'}^{u, \mathfrak{s}} = g_{\bar{t}, \rho}^{u, \mathfrak{s}}$.

2D) We say that $\rho \in \Lambda_k^0$ is explicitly p -reduced when the sequence is with no repetitions and $\langle n(\bar{\ell}_j, \eta_j) : j < j(*) \rangle$ is non-increasing (the length can be zero).

3) For $k < \omega$ let $\Lambda_k^1 = \cup \{ \Lambda_{k, p}^1 : p \in \mathcal{S}^k \}$ where for $p \in \mathcal{S}^k$ we let $\Lambda_{k, p}^1$ be the set of $\rho = \langle (\bar{\ell}_j, \eta_j) : j < j(*) \rangle \in \Lambda_{k, p}^0$ such that: for every \mathfrak{s} and $u \in J^{\mathfrak{s}}$ if $\bar{t} \in {}^k(I_u^{\mathfrak{s}})$ realizes p then there is no $\rho' \in \Lambda_{k, p}^0$ with $\text{supp}(\rho') \subset \text{supp}(\rho)$ and satisfying $g_{\bar{t}, \rho}^{u, \mathfrak{s}} G_u^{<0} = g_{\bar{t}, \rho'}^{u, \mathfrak{s}} G_u^{<0}$.

4) For $k < \omega$ and $p \in \mathcal{S}^k$ let $\Lambda_{k, p}^2$ be the set of finite sequences ϱ of length ≥ 1 such that $\varrho(0) \in \Lambda_{k, p}^0$ and $0 < i < \ell g(\varrho) \Rightarrow \ell g(\varrho(i)) > 0 \wedge \varrho(i) \in \Lambda_{k, p}^1$. Let $\Lambda_k^2 = \cup \{ \Lambda_{k, p}^2 : p \in \mathcal{S}^k \}$.

5) For any \mathfrak{s} , if $u \in J^{\mathfrak{s}}, \bar{t} \in {}^k(I_u)$ and $\varrho = \langle \rho_i : i < i(*) \rangle \in \Lambda_k^2$ then $g_{\bar{t}, \varrho} \in K_u$ (recalling $i(*) \geq 1$) is $g_{\bar{t}, \rho_0} h_{g_{\bar{t}, \rho_1}} h_{g_{\bar{t}, \rho_2}} \dots, h_{g_{\bar{t}, \rho_{i(*)-1}}}$ (product in K_u) where $g_{\bar{t}, \rho_\ell}$ is from Part (2), recalling that $h_g = h_{g_{G_u^{<0}}}$ is from clause (c) of Definition 2.1(2).

5A) For any p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}, \bar{t} \in {}^k(I_u^{\mathfrak{s}})$ and $\varrho = \langle \rho_i : u < i(*) \rangle \in \Lambda_k^2$, let $z_{\bar{t}, \varrho}^u = z_{\bar{t}, \varrho}^{u, \mathfrak{s}} \in Z_u^{2, \mathfrak{s}}$ be $\langle z_{\bar{t}, \rho_i}^u : i < i(*) \rangle$.

5B) For $p \in \mathcal{S}^k$ and $\varrho \in \Lambda_{k, p}^2$ let $\text{supp}(\varrho) = \cup \{ \text{supp}(\rho_i) : i < \ell g(\varrho) \}$.

5C) We say $\varrho \in \Lambda_{k, p}^2$ is p -reduced when for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u^{\mathfrak{s}})$ realizing p , for no $\varrho' \in \Lambda_{k, p}^2$ do we have (in $K_u^{\mathfrak{s}}$) $g_{\bar{t}, \varrho'}^{u, \mathfrak{s}} = g_{\bar{t}, \varrho}^{u, \mathfrak{s}}$ and $\text{supp}(\varrho') \subset \text{supp}(\varrho)$.

2.5 Definition. 1) For $\rho_1, \rho_2 \in \Lambda_{k, p}^0$ we say $\rho_1 \mathcal{E}_{k, p}^0 \rho_2$ or ρ_1, ρ_2 are 0- p -equivalent when: for every p.o.w.i.s. \mathfrak{s} and $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u^{\mathfrak{s}})$ realizing p the elements $g_{\bar{t}, \rho_1}^{u, \mathfrak{s}}, g_{\bar{t}, \rho_2}^{u, \mathfrak{s}}$ of $G_u^{\mathfrak{s}}$ are equal.

2) For $\rho_1, \rho_2 \in \Lambda_{k, p}^1$ we say $\rho_1 \mathcal{E}_{k, p}^2 \rho_2$ or ρ_1, ρ_2 are 1- p -equivalent when: for every

p.o.w.i.s. \mathfrak{s} and $u \in J^\mathfrak{s}$ and $\bar{t} \in {}^k(I_u)$ realizing p we have $g_{\bar{t}, \rho_1}^{u, \mathfrak{s}} G_u^{<0} = g_{\bar{t}, \rho_2}^{u, \mathfrak{s}} G_u^{<0}$.

3) For $\varrho_1, \varrho_2 \in \Lambda_{k,p}^2$ we say that $\varrho_1 \mathcal{E}_{2,p} \varrho_2$ or ϱ_1, ϱ_2 are 2- p -equivalent, when: for every p.o.w.i.s. \mathfrak{s} and $u \in J^\mathfrak{s}$ and $\bar{t} \in {}^k(I_u)$ realizing p the element $g_{\bar{t}, \rho_1}^{u, \mathfrak{s}}$ and $g_{\bar{t}, \rho_2}^{u, \mathfrak{s}}$ of $K_u^\mathfrak{s}$ are equal.

2.6 Claim. 1) In Definition 2.4 parts (2C), (3), (5B) saying “for every p.o.w.i.s. $\mathfrak{s}, u \in J^\mathfrak{s}$ and $\bar{t} \in {}^k(I_u)$ realizing p ” it is equivalent to saying “for some ...”.

2) In Definition 2.4, $E_{k,p}^\iota$ is an equivalence relation on $\Lambda_{k,p}^\iota$ for $\iota = 0, 1, 2$. Every $E_{k,p}^\iota$ -equivalence class contains a reduced member and for $\iota = 0$ even an explicitly reduced one. Explicitly reduced implies reduced.

3) For every p.o.w.i.s. \mathfrak{s} , if $u \in J^\mathfrak{s}$ and $\bar{t} \in {}^k(I_u^\mathfrak{s})$ realizes $p \in \mathcal{S}^k$ then

(a) for $\rho_1, \rho_2 \in \Lambda_{k,p}^0$ we have

$$(\alpha) \quad g_{\bar{t}, \rho_1}^{u, \mathfrak{s}} = g_{\bar{t}, \rho_2}^{u, \mathfrak{s}} \text{ iff } \rho_1 \mathcal{E}_{k,p}^0 \rho_2$$

$$(\beta) \quad \{\ell_{j,i}^{\rho_1} : j < \ell g(\rho_1), i \leq n(\bar{\ell}_j^{\rho_1}, \eta_j^{\rho_1})\} = \{\ell_{j,i}^{\rho_2} : j < \ell g(\rho_1), i \leq n(\bar{\ell}_j^{\rho_1}, \eta_j^{\rho_2})\}$$

$$(\gamma) \quad \text{if } \rho_1, \rho_2 \text{ are explicitly } p\text{-reduced, then they are } \rho_1 \mathcal{E}_{k,p}^0 \rho_2 \text{ iff letting } \rho_i = \langle (\bar{\ell}_j^i, \eta_j^i) : j < j_i \rangle \text{ for } i = 1, 2 \text{ we have}$$

$$(a) \quad j_1 = j_2$$

$$(b) \quad \text{for some permutation } \pi \text{ of } \{0, \dots, j_1 - 1\} \text{ we have} \\ (\bar{\ell}_j^2, \eta_j^2) = (\bar{\ell}_{\pi(j)}^1, \eta_{\pi(j)}^1) \text{ (so actually only the domain of } \mathcal{E}_{0,p} \text{ depends on } p).$$

(b) for $\rho_1, \rho_2 \in \Lambda_{k,p}^1$ we have

$$(\alpha) \quad g_{\bar{t}, \rho_1}^{u, \mathfrak{s}} G_u^{<0} = g_{\bar{t}, \rho_2}^{u, \mathfrak{s}} G_u^{<0} \text{ iff } \rho_1 \mathcal{E}_{k,p}^1 \rho_2$$

4) For every p.o.w.i.s. \mathfrak{s} if $u \in J^\mathfrak{s}$ and $\bar{\ell} \in {}^k(I_u^\mathfrak{s})$ realizes $p \in \mathcal{S}^k$ then

(c) for $\varrho_1, \varrho_2 \in \Lambda_{k,p}^2$ we have

$$(\alpha) \quad g_{\bar{t}, \varrho_1}^{u, \mathfrak{s}} = g_{\bar{t}, \varrho_2}^{u, \mathfrak{s}} \text{ iff } \varrho_1 \mathcal{E}_{k,p}^2 \varrho_2$$

$$(\beta) \quad \text{if } \varrho_1 \mathcal{E}_{k,p}^2 \varrho_2 \text{ and } \varrho_1, \varrho_2 \text{ are } p\text{-reduced then } \text{supp}(\varrho_1) = \text{supp}(\varrho_2).$$

Proof. Straight, (recalling 1.6(7) and note that (3) elaborate (1)).

□_{2.6}

2.7 Claim. Assume $k < \omega, p \in \mathcal{S}^k, \mathfrak{s}$ is a p.o.w.i.s., $u \in J^t$ and $\bar{t}_1, \bar{t}_2 \in {}^k I$ satisfies $p = \text{tp}(\bar{t}_\ell, \emptyset, I_u^{\mathfrak{s}})$ for $\ell = 1, 2$.

- 1) If $\rho \in \Lambda_{k,p}^0$ and ρ is p -reduced and $g_{\bar{t}_1, \rho} = g_{\bar{t}_2, \rho} \in G_u^{\mathfrak{s}}$, then $\bar{t}_2 \restriction \text{supp}(\rho)$ is a permutation of $\bar{t}_1 \restriction \text{supp}(\rho)$.
- 2) If $\rho \in \Lambda_{k,p}^1$ is p -reduced and $g_{\bar{t}_1, \rho}^{u, \mathfrak{s}} G_u^{<0} = g_{\bar{t}_2, \rho}^{u, \mathfrak{s}} G_u^{<0}$ then $\bar{t}_1 \restriction \text{supp}(\rho)$ is a permutation of $\bar{t}_2 \restriction \text{supp}(\rho)$.
- 3) If $\varrho \in \Lambda_{k,p}^2$ is p -reduced and $g_{\bar{t}_1, \varrho}^{u, \mathfrak{s}} = g_{\bar{t}_2, \varrho}^{u, \mathfrak{s}}$ so both are well defined then similarly $\bar{t}_1 \restriction \text{supp}(\varrho)$ is a permutation of $\bar{t}_2 \restriction \text{supp}(\varrho)$ and both are with no repetition.
- 4) For every $\varrho_1 \in \Lambda_{k,p}^2$ there is a p -reduced ϱ_2 such that for every p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u^{\mathfrak{s}})$ realizing p we have $g_{\bar{t}, \varrho_1}^{u, \mathfrak{s}} = g_{\bar{t}, \varrho_2}^{u, \mathfrak{s}}$. (Similarly for $\Lambda_{k,p}^0, \Lambda_{k,p}^1$).

Proof. Straight.

2.8 Definition. Let \mathfrak{s} be a κ -p.o.w.i.s.

- 1) For $u \leq_{J[\mathfrak{s}]} v$ let $\hat{\pi}_{u,v}^0$ be the following partial mapping from $Z_v^{0, \mathfrak{s}}$ to $Z_u^{0, \mathfrak{s}}$, recalling Definition 2.1(3)(b):

$x \in \text{Dom}(\hat{\pi}_{u,v}^0)$ iff $x \in Z_v^{0, \mathfrak{s}}$ and $\pi_{u,v}(t_\ell^x)$ is well defined for $\ell \leq n(x)$ and then $\hat{\pi}_{u,v}(x) = (\langle \pi_{u,v}(t_\ell^x) : \ell \leq n(x) \rangle, \eta^x)$.

- 2) For $u \leq_{J[\mathfrak{s}]} v$ let $\hat{\pi}_{u,v}^1 = \hat{\pi}_{u,v}^{1, \mathfrak{s}}$ be the following partial mapping Z_v^1 to Z_u^1 : if $z \in Z_u^1$ so $z = (\langle (\bar{t}^k, \eta^k) : k < k^* \rangle$ and $\bar{t}^k = \langle t_\ell^k : \ell \leq \ell_k \rangle, t_\ell^k \in I_v$ for $k < k^*, \ell \leq \ell_k$ then $\hat{\pi}_{u,v}^1(z) = (\langle (\langle \pi_{u,v}(t_\ell^k) : \ell < \ell_k \rangle, \eta^k) : k < k^* \rangle$ when each $\pi_{u,v}(t_\ell^k)$ is well defined.

- 3) Let $u \leq_{J[\mathfrak{s}]} v$ let $\hat{\pi}_{u,v}$ be $\hat{\pi}_{u,v}^0 \cup \hat{\pi}_{u,v}^1$.

- 4) For $u \in J^{\mathfrak{s}}$ and $z \in Z_u$ let $\partial_{u,z}$ be the following permutation of $D_u = D_u^{\mathfrak{s}}$ where D_u is from Definition 2.1(3)(a).

For each $(v, g) \in D_u$ we define $\partial_{u,z}((v, g))$ as follows:

Case 1: $z \in \text{Dom}(\hat{\pi}_{v,u}^0) \subseteq Z_u^0$ and $\hat{\pi}_{v,u}(z) \in X_v^{\mathfrak{s}}$, i.e., $\langle \hat{\pi}_{v,u}(t_\ell^z) : \ell \leq n(*) \rangle$ is \leq_{I_u} -decreasing.

Then let $\partial_{u,z}((v, g)) = (v, g_{\hat{\pi}_{v,u}(z)} g)$ noting $g_{\pi_{v,u}(z)} \in G_v \subseteq K_v$.

Case 2: $z \in \text{Dom}(\hat{\pi}_{v,u}^1) \subseteq Z_u^1$ so $z = \langle x_\ell : \ell < k \rangle$ and $x_\ell \in \text{Dom}(\hat{\pi}_{v,u}^0)$ for $\ell < k$ and let $x'_\ell := \hat{\pi}_{v,u}^0(x_\ell) \in X_v^{\mathfrak{s}}$ for $\ell < k$.

Then let $\partial_{u,z}((v, g)) = (v, g')$ when $g' \in K_v$ is defined by as $h_{g_{x'_0} \dots g_{x'_{k-1}}} g$, product in K_u noting $g_{x'_\ell} \in G_v \subseteq K_v$ for $\ell < k$.

Case 3: Neither case 1 nor case 2.

Then let $\partial_{u,x}((v, g)) = (v, g)$.

2.9 Observation. In Definitions 2.1, 2.8:

- 1) If $u \leq_{J[\mathfrak{s}]} v$ then $\hat{\pi}_{u,v}$ is a partial mapping from Z_v to Z_u .
- 2) In part (1), $\hat{\pi}_{u,v}$ maps Z_v^0, Z_v^1 to Z_u^0, Z_u^1 respectively, that is it maps $Z_v^\ell \cap \text{Dom}(\hat{\pi}_{u,v})$ into Z_u^ℓ for $\ell = 0, 1$.
- 3) If $u \leq_{J[\mathfrak{s}]} v$ and \mathfrak{s} is nice or just $\text{Dom}(\pi_{u,v}) = I_v$ then $\text{Dom}(\hat{\pi}_{u,v}) = Z_v$.
- 4) $\text{nor}_{K_u}(H_u)$ is $K_u^{<0}$ where H_u is from Definition 2.1(1)(f).
- 5) $\text{nor}_{K_u}^{1+\alpha}(H_u)$ is $K_u^{<\alpha}$ for $\alpha \geq 0$ if \mathfrak{s} is non-trivial.

Proof. 1),2),3) Check.

4) As H_u has two elements e_{K_u} and h_* clearly an element of K_u normalize H_u iff it commutes with g_* . Now when does $(g, h) \in G_u *_{\mathbf{h}_u} L_u$ commute with $g_* = (e_{G_u}, h_{e_{G_u} G_u^{<0}})$? Note that

$$(g, h)(e_{G_u}, h_{e_{G_u} G_u^{<0}}) = (g, h + h_{e_{G_u} G_u^{<0}})$$

$$(e_{G_u}, h_{e_{G_u} G_u^{<0}})(g, h) = (g, ((\mathbf{h}_u(g))(h_{e_{G_u} G_u^{<0}}) + h).$$

As L_u is commutative, “ (g, h) commute in K_u ” iff in L_u

$$(\mathbf{h}_u(g))(h_{e_{G_u} G_u^{<0}}) = h_{e_{G_u} G_u^{<0}}.$$

By the definition of $\mathbf{h}_u \in \text{Hom}(G_u, \text{Aut}(L_u))$ in 2.1(1)(d),(e) this means

$$(ge_{G_u})G_u^{<0} = e_{G_u}G_u^{<0}.$$

i.e.

$$g \in G_u^{<0}.$$

We can sum that: $(g, h) \in G_u *_{\mathbf{h}_u} L_u$ belongs to $\text{nor}_{K_u}(H_u)$ iff (g, h) commutes with h_* iff $g \in G_u^{<0}$ iff $(g, h) \in K_u^{<0}$, as required.

5) Let $\mathbf{f}_u : K_u \rightarrow G_u$ be defined by $\mathbf{f}_u((g, h)) = g$. Clearly

- (*)₁ \mathbf{f}_u is a homomorphism from K_u onto G_u and for every ordinal $\alpha \geq 0$, it maps $K_u^{<\alpha}$ onto $G_u^{<\alpha}$ so $\mathbf{f}_u(K_u^{<\alpha}) = G_u^{<\alpha}$ and moreover $\mathbf{f}^{-1}(G_u^{<\alpha}) = K_u^{<\alpha}$ (see the definition of $K_u^{<\alpha}$ in 2.1(2)).

Also

(*)₂ $\text{Ker}(\mathbf{f}_u) = \{e_{G_u}\} \times L_u \subseteq K_u^{<0}.$

Now we prove by induction on the ordinal $\alpha \geq 0$ that $\text{nor}_{K_u}^{1+\alpha}(H_u) = K_u^{<\alpha}$. For $\alpha = 0$ this holds by part (4). For α limit this holds as both $\langle \text{nor}_{K_u}^\beta(H_u) : \beta \leq \alpha \rangle$ and $\langle K_u^{<\beta} : \beta \leq \alpha \rangle$ are increasing continuous.

Lastly, for $\alpha = \beta + 1 > 0$ we have for any $f \in K_u$

$$\begin{aligned}
f \in \text{nor}_{K_u}^{1+\alpha}(H_\beta) &\Leftrightarrow f \in \text{nor}_{K_u}(\text{nor}_{K_u}^{1+\beta}(H_\beta)) \\
&\Leftrightarrow f \in \text{nor}_{K_u}(\mathbf{f}_u^{-1}(G_u^{<\beta})) \\
&\Leftrightarrow f(\mathbf{f}_u^{-1}(G_u^{<\beta}))f^{-1} = \mathbf{f}_u^{-1}(G_u^{<\beta}) \\
&\Leftrightarrow \mathbf{f}_u(f)G_u^{<\beta}\mathbf{f}_u(f)^{-1} = G_u^{<\beta} \\
&\Leftrightarrow \mathbf{f}_u(f) \in \text{nor}_{G_u}(G_u^{<\beta}) \\
&\Leftrightarrow \mathbf{f}_u(f) \in G_u^{<\alpha} \Leftrightarrow f \in K_u^{<\alpha}.
\end{aligned}$$

[Why? The first \Leftrightarrow by the definition of $\text{nor}_{K_u}^{\beta+1}(-)$, the second \Leftrightarrow by the induction hypothesis, the third \Leftrightarrow by the definition of $\text{nor}_{K_u}(-)$, the fourth \Leftrightarrow by $(*)_1$, the fifth \Leftrightarrow by the definition of $\text{nor}_{G_u}(-)$, the sixth \Leftrightarrow by 1.9(1), the seventh \Leftrightarrow by $(*)_1$.] $\square_{2.9}$

2.10 Observation. Let \mathfrak{s} be a p.o.w.i.s.

- 1) For $u \in J^\mathfrak{s}$ and $x \in Z_u^\mathfrak{s}$ we have: $\partial_{u,x}$ is a well defined function and is a permutation of $D_u^\mathfrak{s}$.
- 2) If $u \leq_{J[\mathfrak{s}]} v$ then $D_u^\mathfrak{s} \subseteq D_v^\mathfrak{s}$.
- 3) If $u \leq_{J[\mathfrak{s}]} v$ and $y \in Z_v^\mathfrak{s}$ and $x = \hat{\pi}_{u,v}(y)$ then $\partial_{u,x} = \partial_{v,y} \upharpoonright D_u$.
- 4) If \mathfrak{s} is nice and $u \in J^\mathfrak{s}$ and $z \in Z_u^\mathfrak{s}$ then in the definition 2.8(4) of $\partial_{u,z}$ Case 3 never occurs.

Proof. Straight.

2.11 Definition. Let \mathfrak{s} be κ -p.o.w.i.s.

- 1) Let $\mathbf{S}^k = \{\mathbf{q} : \mathbf{q} \text{ is a function with domain } \mathcal{S}^k \text{ and for } q \in \mathcal{S}^k, \mathbf{q}(p) \in \Lambda_{k,p}^2\}$, on $\Lambda_{k,p}^2$, see Definition 2.4(4) above.
- 2) We say that $\mathbf{q} \in \mathbf{S}^k$ is disjoint when $\langle \text{supp}(\mathbf{q}(p)) : p \in \mathcal{S}^k \rangle$ is a sequence of pairwise disjoint sets. We say that \mathbf{q} is reduced when $\mathbf{q}(p)$ is p -reduced for every $p \in \mathcal{S}^k$.
- 3) Let $Z_u^2 = Z_u^{2,\mathfrak{s}}$ be $\cup\{Z_u^{2,k} : k < \omega\}$, where $Z_u^{2,k} = Z_u^{2,k,\mathfrak{s}}$ is the set of pairs (\bar{t}, \mathbf{q}) where for some $k < \omega$, $\bar{t} \in {}^k(I_u^\mathfrak{s})$ and $\mathbf{q} \in \mathbf{S}^k$.

- 4) For $z = (\bar{t}, \mathbf{q}) \in Z_u^2$ let $\partial_{u,z} = \partial_{u,z}^s$ be the following permutation of D_u : if $v \leq_{J[\bar{t}]} u$ and $(v, g) \in \{v\} \times K_v$ then $\partial_{u,z}^s((v, g)) = (v, g'g)$ where $g' = g_{\pi_{v,u}(\bar{t}), \mathbf{q}(p)}^{v,s}$ where $p = \text{tp}_{\text{qf}}(\pi_{v,u}(\bar{t}), \emptyset, I_v^s)$, and, of course, $\pi_{v,u}(\langle t_\ell : \ell < k \rangle) = \langle \pi_{v,u}(t_\ell) : \ell < k \rangle$.
- 5) For $(\bar{t}, \mathbf{q}) \in Z_u^2$ let $g_{\bar{t}, \mathbf{q}} = g_{\bar{t}, \mathbf{q}}^u = g_{\bar{t}, \mathbf{q}}^{u,s} = g_{\bar{t}, \mathbf{q}(p)}$ when $p = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, I_u)$. Let $g_{\bar{t}, \mathbf{q}}^{v,s} = g_{\bar{t}, \mathbf{q}}^{v,s} = g_{\pi_{v,u}(\bar{t}), \mathbf{q}}^v$ when $v \leq_{J[\bar{s}]} u$.

2.12 Remark. We can add $\{\partial_{u,z}^s : z \in Z_u^{2,s}\}$ to the generators of F_u^s defined in 2.14 below.

2.13 Observation. In Definition 2.11(2), $\partial_{u,z}^s$ is a well defined permutation of D_u^s .

Proof. Easy.

2.14 Definition. Let \mathfrak{s} be a p.o.w.i.s.

1) Let $F_u = F_u^s$ be the subgroup of the group of permutations of D_u^s generated by $\{\partial_{u,z} : z \in Z_u^s\}$.

2) For a p.o.w.i.s. \mathfrak{s} let M_s be the following model:

set of elements: $\{(u, g) : u \in J^s \text{ and } g \in K_u^s\} \cup \{(1, u, f) : u \in J^s \text{ and } f \in F_u^s\}$.

relations: $P_{1,u}^{M_s}$, a unary relation, is $\{(u, g) : g \in K_u\}$ for $u \in J^s$,

$P_{2,u}^{M_s}$, a unary relation is $\{(1, u, f) : f \in F_u\}$ for $u \in J^s$

$R_{u,v,h}^{M_s}$, a binary relation, is $\{((v, g), (1, u, f)) : f \in F_u, g \in K_v \text{ and } f((v, h)) = (v, g)\}$ for $u \in J^s$ and $v \leq_{J[\bar{s}]} u$ and $h \in K_v$.

2.15 Observation. If \mathfrak{s} is a κ -p.o.w.i.s. and $v \leq_{J[\bar{s}]} u$ and $f \in F_u$ then f maps $\{v\} \times K_v = P_{1,v}^{M_s}$ onto itself.

Remark. If $\pi \in F_u^s$ and $v \leq_{I_u^s[\bar{s}]} u$ then $\pi \upharpoonright (\{v\} \times K_v)$ comes directly from K_v^s , but the relation between the $\langle \pi \upharpoonright (\{v\} \times K_v) : v \leq_{I_u[\bar{s}]} u \rangle$ are less clear.

2.16 Claim. Let \mathfrak{s} is a p.o.w.i.s.

1) \varkappa is an automorphism of M_s iff:

- ⊗ (a) \varkappa is a function with domain M_s
- (b) for every $u \in J^s$ we have:
 - (α) $\varkappa \upharpoonright D_u \in F_u^s$ for every $u \in J^s$
 - (β) letting $f_u = \varkappa \upharpoonright D_u$ we have $(1, u, f) \in P_{2,u}^{M_s} \Rightarrow \varkappa((1, u, f)) = (1, u, f_u f)$ where $f_u f$ is the product in F_u .

2) If $f_u \in F_u$ for $u \in J^s$ and $f_u \subseteq f_v$ for $u \leq_{J[s]} v$ then there is one and only one automorphism κ of M_s such that $u \in J^s \Rightarrow f_u \subseteq \kappa$.

Proof. First assume that $\bar{f} = \langle f_u : u \in J^s \rangle$ is as in part (2). We define $\kappa_{\bar{f}}$, a function with domain M_s by:

- ⊗₁ (a) if $a = (u, g) \in P_{1,u}^{M_s}$ and $u \in J^s$ then $\kappa_{\bar{f}}(a) = f_u(a)$
- (b) if $a = (1, u, f) \in P_{2,u}^{M_s}$ then $\kappa_{\bar{f}}(a) = (1, u, f_u f)$.

So

- ⊗₂ (a) $\kappa_{\bar{f}}$ is a well defined function
- (b) $\kappa_{\bar{f}}$ is one to one
- (c) $\kappa_{\bar{f}}$ is onto M_s
- (d) $\kappa_{\bar{f}}$ maps $P_{1,u}^{M_s}$ onto $P_{1,u}^{M_s}$ and $P_{2,u}^{M_s}$ onto $P_{2,u}^{M_s}$ for $u \in J^s$
- (e) also $\bar{f}' = \langle f_u^{-1} : u \in J^s \rangle$ satisfies the condition of part (2) and $\kappa_{\bar{f}'}$ is the inverse of $\kappa_{\bar{f}}$
- (f) $\kappa_{\bar{f}}$ maps $R_{u,v,h}^{M_s}$ onto itself.

[Why? The only non-trivial one is clause (f) and in it by clause (e) it is enough to prove that $\kappa_{\bar{f}}$ maps $R_{u,v,h}^{M_s}$ into $R_{u,v,h}^{M_s}$. So assume $v \leq_{J[s]} u, h \in K_v$ and $((v, g), (1, u, f)) \in R_{u,v,h}^{M_s}$ hence $f \in F_u, g \in K_v$ and $f((v, h)) = (v, g)$. So $\kappa_{\bar{f}}((v, g)) = f_v((v, g))$ and $\kappa_{\bar{f}}(1, u, f) = (1, u, f_u f)$ and we would like to show that $(f_v((v, g)), (1, u, f_u f)) \in R_{u,v,h}^{M_s}$.

This means that $(f_u f)((v, h)) = f_v((v, g))$. We know that $f((v, h)) = (v, g)$ hence $(f_u f)((v, h)) = f_u(f((v, h))) = f_u((v, g))$ so we have to show that $f_u((v, g)) = f_v((v, g))$. But $v \leq_{J[s]} u$ hence (by the assumption on \bar{f}) we have $f_u \subseteq f_v$ hence $f_u((v, g)) = f_v((v, g))$ so we are done.]

So we have shown that

- ⊗₃ if $\bar{f} = \langle f_u : u \in J^s \rangle$ is as in part (2) then $\kappa_{\bar{f}}$ is an automorphism of M_s .

Next

- ⊗₄ if $\kappa \in \text{Aut}(M_s)$ and $\kappa \upharpoonright P_{1,u}^{M_s}$ is the identity for each $u \in J^s$ then $\kappa = \text{id}_{M_s}$.

[Why? By the $R_{u,v,h}^{M_s}$'s and F_u^s being a group of permutations of D_u .]

- ⊗₅ the mapping $\varkappa \mapsto \langle \varkappa \upharpoonright P_{1,u}^{M_s} : u \in J^s \rangle$ is a homomorphism from $\text{Aut}(M_s)$ into $\{\varkappa_{\bar{f}} : \bar{f} \text{ as above}\}$ with coordinatewise product, with kernel $\{\varkappa \in \text{Aut}(M_s) : \varkappa \upharpoonright P_{1,u}^{M_s} = \text{id}_{P_{1,u}^{M_s}} \text{ for every } u \in J^s\}$.

[Why? Easy.]

- ⊗₆ the mapping above is onto.

[Why? Given $\varkappa \in \text{Aut}(M_s)$, let $f_u = \varkappa \upharpoonright P_{1,u}^{M_s}$. Clearly $f_u \in F_u$ and $u \leq_{J[s]} v \Rightarrow f_u \subseteq f_v$ so $\bar{f} = \langle f_u : u \in J^s \rangle$ is as above so by ⊗₃ we know $\varkappa_{\bar{f}}$ is an automorphism of M_s and $\varkappa_{\bar{f}} \varkappa^{-1}$ is an automorphism of M_s which is the identity on each $P_{1,u}^{M_s}$ hence by ⊗₄ is id_{M_s} . So $\varkappa = \varkappa_{\bar{f}}$, is as required.]

- ⊗₇ the mapping above is one to one.

[Why? Easy by ⊗₄.]

Together both parts should be clear. □_{2.16}

2.17 Definition. 1) We say that $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ are \mathcal{S} -equivalence where $\mathcal{S} \subseteq \mathcal{S}^k$ when $p \in \mathcal{S} \Rightarrow \mathbf{q}_1(p) \mathcal{C}_{2,p} \mathbf{q}_2(p)$.
2) Omitting \mathcal{S} means $\mathcal{S} = \mathcal{S}^k$.

2.18 Claim. 1) If $u \in J^s$ and $f \in F_u^s$ then for some k and $\bar{t} = \langle \bar{t}_\ell : \ell < k \rangle \in {}^k(I_u^s)$ and $\mathbf{q} \in \mathbf{S}^k$ we have:

- (*) $f = \partial_{u,(\bar{t},\mathbf{q})}$ (so if $v \leq_{J[s]} u$ then $f \upharpoonright (\{v\} \times K_v^s)$ is moving by multiplication by $g_{(\pi_{v,u}(\bar{t}),\mathbf{q})}$, e.g. $g \in K_v \Rightarrow f((v,g)) = (v, g_{\pi_{v,u}(\bar{t}),\mathbf{q}})$).

2) $\{\partial_{u,(\bar{t},\mathbf{q})} : \bar{t} \in {}^k(I_u^s) \text{ and } \mathbf{q} \in \mathbf{S}^k \text{ for some } k\}$ is a group of permutations of D_u^s which include F_u^s .

3) For every $\mathbf{q} \in \mathbf{S}^k$ there is a reduced $\mathbf{q}' \in \mathbf{S}^k$ which is \mathbf{S}^k -equivalent to it (see Definition 2.11(2)).

2.19 Remark. 1) We can be somewhat more restrictive.

Proof. We use freely Definition 2.11. Recall that F_u^s is the group of permutations of D_u^s generated by $\{\partial_{u,z} : z \in Z_u^s\}$. Hence it is enough to prove that $f \in F_u^s$ satisfies the conclusion of the claim in the following cases.

Case 0: f is the identity.

It is enough to let $k = 0$ so \mathcal{S}^k is a singleton $\{p\}$ and $\mathbf{q}(p)$ is the sequence $\langle\langle\rangle\rangle$, i.e. we use in Definition 2.4(1) the case $j(*) = 0$, i.e. 2.1(3) for $k = 0$.

Case 1: $f = \partial_{u,z}$ where $z \in Z_u^0$.

So $z = X_{I_u}$ let $k = n(z) + 1, \bar{t} = \bar{t}^z$. We define \mathbf{q} as follows:

- (a) if $q \in \mathcal{S}^k$ “says” that $\bar{t} = \langle t_\ell : \ell \leq n(z) \rangle$ is decreasing then $g_{\bar{t}, \mathbf{q}}$ is g_z
- (b) if not then $g_{\bar{t}^z, \mathbf{q}} = e_{K_u}$.

Case 2: $f = \partial_{u,z}$ where $z \in Z_u^1$.

Also clear.

Case 3: $f = f_1 f_2$ (product in F_u^s) where $f_1, f_2 \in F_u^s$ satisfies the conclusion of the claim.

Just combine the definitions.

Case 4: $f = f^{-1}$ where $f \in F_u^s$ satisfies the conclusion of the claim.

Easy, too.

□_{2.18}

2.20 Remark. If $q \in \mathcal{S}^k$ and $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ and $v \leq_{J[\mathfrak{s}]} u, \bar{\mathbf{t}} = {}^k(I_u)$ and $q = \text{tp}_{\text{qf}}(\pi_{v,u}^{\mathfrak{s}}(\bar{t}), \emptyset, I_v)$ and $\mathbf{q}_1(q), \mathbf{q}_2(q)$ are not $\mathcal{E}_{2,q}$ -equivalent, then $g_{\bar{\mathbf{t}}, \mathbf{q}_1} \neq g_{\bar{\mathbf{t}}, \mathbf{q}_2}$.

Proof. This is by 2.4(4).

§3 THE MAIN RESULT

We can prove that every κ -parameter has a limit, but for our application it is more transparent to consider κ -parameter \mathfrak{s} which is the κ -parameter \mathfrak{t} + its limit.

3.1 Definition. We say that \mathfrak{s} is the limit of \mathfrak{t} as witnessed by v_* when (both are p.o.w.i.s. and)

- (a) $J^{\mathfrak{t}} \subseteq J^{\mathfrak{s}}$ and $J^{\mathfrak{s}} = J^{\mathfrak{t}} \cup \{v_*\}$, $v_* \notin J^{\mathfrak{t}}$ and $u \in J^{\mathfrak{s}} \Rightarrow u \leq_{J[\mathfrak{s}]} v_*$
- (b) $I_u^{\mathfrak{s}} = I_u^{\mathfrak{t}}$ and $\pi_{u,v}^{\mathfrak{s}} = \pi_{u,v}^{\mathfrak{t}}$ when $u \leq_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_*$
- (c) if $t \in I_{v_*}^{\mathfrak{s}}$ then for some $u = u_t \in J^{\mathfrak{s}}$ we have $t \in \text{Dom}(\pi_{u_t, v_*}^{\mathfrak{s}})$, moreover (if \mathfrak{s} is nice this follows) $J^{\mathfrak{s}} \models "u_t \leq v < v_*" \Rightarrow t \in \text{Dom}(\pi_{v, v_*}^{\mathfrak{s}})$
- (d) if $s, t \in I_{v_*}^{\mathfrak{s}}$ then for some $u = u_{s,t} \in J^{\mathfrak{t}}$ for every v satisfying $u \leq_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_*$ we have $I_{v_*}^{\mathfrak{s}} \models "s < t" \Leftrightarrow \pi_{v, v_*}^{\mathfrak{s}}(s) <_{I_v^{\mathfrak{s}}} \pi_{v, v_*}^{\mathfrak{s}}(t)$
- (e) if $\langle t_u : u \in J_{\geq w}^{\mathfrak{t}} \rangle$ is a sequence satisfying $w \in J$, $J_{\geq w} = \{u : w \leq u \in J\}$; $t_u \in I_u^{\mathfrak{s}}$ and $w \leq u_1 \leq u_2 \in J^{\mathfrak{t}} \Rightarrow \pi_{u_1, u_2}(t_{u_2}) = t_{u_1}$, then there is a unique $t \in I_{v_*}^{\mathfrak{s}}$ such that $u \in J_{\geq w}^{\mathfrak{t}} \Rightarrow \pi_{u, v_*}(t) = t_u$.

3.2 Definition. We say that \mathfrak{s} is an existential limit of \mathfrak{t} when: clauses (a)-(e) of Definition 3.1 holds and

- (f) assume that
 - (α) $u_* \in J^{\mathfrak{t}}$
 - (β) $k_1, k_2 < \omega$ and $k = k_1 + k_2$
 - (γ) E is an equivalence relation on \mathcal{S}^k
 - (δ) $\bar{e} = \langle e_u : u \in J_{\geq u_*}^{\mathfrak{t}} \rangle$, where e_u is an E -equivalence class
 - (ε) $\bar{t} \in {}^{k_1}(I_{v_*}^{\mathfrak{s}})$
 - (ζ) for every $v \in J_{\geq u_*}^{\mathfrak{t}}$ there is $\bar{s}_v \in {}^{k_2}(I_{w(v)}^{\mathfrak{t}})$ such that:
if $u_* \leq_{J[\mathfrak{t}]} u \leq_{J[\mathfrak{t}]} v$ then e_u is the E -equivalence class of $\text{tp}_{\text{qf}}(\bar{t}^u \wedge \bar{s}^{u,v}, \emptyset, I_u^{\mathfrak{t}})$ where $\bar{t}^u = \pi_{u, v_*}^{\mathfrak{s}}(\bar{t})$ and $\bar{s}^{u,v} = \pi_{u, v}^{\mathfrak{t}}(\bar{s}_v)$.

Then there is $\bar{s} \in {}^{k^*}(I_{v_*}^{\mathfrak{s}})$ such that for every $u \in J^{\mathfrak{t}}$ large enough $\text{tp}(\pi_{u, v_*}^{\mathfrak{s}}(\bar{t} \wedge \bar{s}), \emptyset, I_u^{\mathfrak{t}})$ belongs to e_u (and is constantly p_* for some $p_* \in \mathcal{S}^k$).

3.3 Remark. We may say “ \mathfrak{s} is semi-limit of \mathfrak{t} ” when in clause (d) we replace \Leftrightarrow by \Rightarrow . We may consider using this weaker version and/or omit linearity in our main theorem, but the present version suffices.

3.4 Main Claim. $K_{v_*}^{\mathfrak{s}}$ is an almost κ -automorphism group (see below) when:

- ⊠ (a) $\mathfrak{s}, \mathfrak{t}$ are both p.o.w.i.s
- (b) \mathfrak{s} is an existential limit of \mathfrak{t} as witnessed by v_*
- (c) $J^{\mathfrak{t}}$ is \aleph_1 -directed and is linear (i.e., for every $u, v \in J^{\mathfrak{t}}$ we have $u \leq_{J[\mathfrak{t}]} v$ or $v \leq_{J[\mathfrak{t}]} u$)
- (d) \mathfrak{t} is a κ -p.o.w.i.s (so $\kappa \geq |J^{\mathfrak{t}}|$ and $\kappa \geq |I_u^{\mathfrak{t}}|$ for $u \in J^{\mathfrak{t}}$)
- (e) \mathfrak{t} is non-trivial (see Definition 1.1(6)).

Remark. Not much harm in adding \mathfrak{t} is nice (see Definition 1.2(7)) so for $u \leq_{J[\mathfrak{t}]} v$ the functions $\pi_{u,v}^{\mathfrak{t}}, \hat{\pi}_{u,v}^{\mathfrak{t}}$ has full domain, see Definition 2.8(1),(2),(3) and Claim 2.9(3)).

3.5 Definition. G is an almost κ -automorphism group when: there is a κ -automorphism group G^+ and a normal subgroup G^- of G^+ of cardinality $\leq \kappa$ such that G is isomorphic to G^+/G^- , i.e., there is a homomorphism from G^+ onto G with kernel G^- .

Before proving 3.4 we explain: why being almost κ -automorphism group help us in proving our intended result?

Recalling 0.7:

3.6 Claim. For any ordinal α , if there is an almost κ -automorphism group G with a subgroup H of cardinality $\leq \kappa$ such that $\tau'_{G,H} = \alpha$ [such that $\text{nor}_G^\alpha(H) = G \wedge (\forall \beta < \alpha)(\text{nor}_G^\beta(H) \neq G)$] then there is a κ -automorphism group G' with a subgroup H' of cardinality $\leq \kappa$ such that $\tau'_{G',H'} = \alpha$ [such that $\text{nor}_{G'}^\alpha(H') = G' \wedge (\forall \beta < \alpha)(\text{nor}_{G'}^\beta(H') \neq G')$].

Proof. Easy.

Let G^+, G^- be as in Definition 3.5 and h be a homomorphism from G^+ onto G with kernel G^- and let $H^+ = \{x \in G^+ : h(x) \in H\}$.

So it is easy to check each of the following statements (similar to 2.9(5)):

- ⊗ (a) H^+ is a subgroup of G^+
- (b) $|H^+| \leq |H| \times |G^-| \leq \kappa\kappa = \kappa$
- (c) G^+ is a κ -automorphism group
- (d) $\text{nor}_{G^+}^\beta(H^+) = \{x \in G^+ : h(x) \in \text{nor}_G^\beta(H)\}$ for every $\beta \leq \infty$
- (e) $\tau_{G,H} = \tau_{G^+,H^+}$
- (f) $\text{nor}_G^\beta(H) = G$ then $\text{nor}_{G^+}^\beta(H^+) = G^+$ for every $\beta \leq \infty$.

Together (G^+, H^+) exemplifies the desired conclusion. $\square_{3.6}$

Proof of 3.4. Let G^+ be the automorphism group of $M_{\mathfrak{t}}$ and let G^- be the following subgroup of G^+

$$\{\varkappa \in G^+ : \text{for some } u \in J^{\mathfrak{t}} \text{ we have} \\ u \leq_J v \wedge g \in K_v \Rightarrow \varkappa((v, g)) = (v, g)\}.$$

Easily

- \otimes_1 G^- is a subgroup of G^+
 [Why? As $J^{\mathfrak{t}}$ is directed]
- \otimes_2 for every $\varkappa \in G^+$ we can find $\bar{f}^\varkappa = \langle f_u^\varkappa : u \in J^{\mathfrak{t}} \rangle$ such that

 - (a) $f_u^\varkappa \in F_u^{\mathfrak{t}}$
 - (b) $\varkappa \upharpoonright D_u^{\mathfrak{t}} = f_u$
 - (c) $\varkappa \upharpoonright P_{2,u}^{M_{\mathfrak{t}}}$ is $(1, u, f) \mapsto (1, u, f_u f)$.
 [Why? By Claim 2.16.]
- \otimes_3 G^- has cardinality $\leq \kappa$.
 [Why? As $|J^{\mathfrak{t}}| \leq \kappa$, it suffices to prove that for each $u \in J^{\mathfrak{t}}$, the subgroup $G_u^- := \{\varkappa \in G^+ : \varkappa \upharpoonright P_{1,v}^{M_{\mathfrak{t}}}$ is the identity when $u \leq_{J[\mathfrak{s}]} v\}$ has cardinality $\leq \kappa$, but this has the same number of elements as $F_u^{\mathfrak{s}}$ because $\varkappa \mapsto \varkappa \upharpoonright D_u$ is a one-to-one function from G_u^- onto $F_u^{\mathfrak{s}}$ and \mathfrak{t} is linear. As $|F_u^{\mathfrak{s}}| \leq \aleph_0 + |Z_u| = \aleph_0 + |I_u| \leq \kappa$ we are done.]
- \otimes_4 G^- is a normal subgroup of G^+ .
 [Why? By its definition, more elaborately

 - (a) each G_u^- is a normal subgroup of G^+ .
 [Why? As all members of $\text{Aut}(M_{\mathfrak{s}})$ maps each $\{v\} \times K_v$ onto itself so G_u^- is even an definable subgroup]
 - (b) $u \leq_{J[\mathfrak{t}]} v \Rightarrow G_u^- \subseteq G_v^-$.
 [Why? Check the definitions.]
 - (c) $G^- = \cup \{G_u^- : u \in J\}$.
 [Why? Trivially.]

Together we are done proving \otimes_4 .]

- \otimes_5 For $x \in Z_{v^*}^5$ let \varkappa_x be the following automorphism of M_t , it is defined as in \otimes_2 by $\langle f_u^x : u \in J^t \rangle$ where $f_u = \partial_{u, \hat{\pi}_{u, v^*}(x)}^t$ is from Definition 2.8(4)
- \otimes_6 for every $x \in Z_{v^*}^5$, \varkappa_x is a well defined automorphism of M_t .
[Why? Look at the definitions and 2.16.]

The main point is

- \otimes_7 G^+ is generated by $\{\varkappa_x : x \in Z_{v^*}^5\} \cup G^-$.

Why? Clearly the set is a set of elements of G^+ . So assume $\varkappa \in G^+$ and let $\bar{f}^\varkappa = \langle f_u^\varkappa : u \in J^t \rangle$ be as in \otimes_2 , they are fixed for awhile.

By 2.18 for each $u \in J^t$ there are $k = k^u$ and $\bar{t} = \bar{t}^u \in {}^{k^u}(I_u^5)$ and $\mathbf{q} = \mathbf{q}^u \in \mathbf{S}^{k^u}$ such that (the “disjoint” as we can replace \bar{t} by $\bar{t} \wedge \bar{t}$ or even $\bar{t} \wedge \bar{t} \wedge \dots \wedge \bar{t}$ with $|S^{k^u}|$ copies note that we can demand that \mathbf{q} is reduced by 2.18(3)):

- \boxminus_1 $f_u^\varkappa = \partial_{u, (\bar{t}^u, \mathbf{q})}$, i.e., if $v \leq_{J[t]} u$ then $f \upharpoonright (\{v\} \cap K_v^t)$ is a multiplication from the left (of the K_v^t -coordinate) by $g_{\pi_{v, u}^t(\bar{t}), \mathbf{q}^u}$ and \mathbf{q}^u is reduced and disjoint, see Definition 2.11(2),(5).

The choices are not necessarily unique, in particular

- \boxminus_2 if $u^1 \leq_{J[t]} u^2$ then $(k^{u^2}, \pi_{u^1, u^2}(\bar{t}^{u^2}), \mathbf{q}^{u^2})$ can serve as $(k^{u^1}, \bar{t}^{u^1}, \mathbf{q}^{u^1})$.

Also

- \boxminus_3 the set of possible (k^u, \mathbf{q}^u) is countable.

As J^t is \aleph_1 -directed

- \boxminus_4 for some pair (k^*, \mathbf{q}^*) the set $\{u \in J^t : k^u = k^* \text{ and } \mathbf{q}^u = \mathbf{q}^*\}$ is cofinal in J^t .

Together, without loss of generality for some k^*, \mathbf{q}

- \boxminus_5 $k^u = k^*$ and $\mathbf{q}^u = \mathbf{q}$ for every $u \in J^t$.

Let E be an ultrafilter on J^t such that $u \in J^t \Rightarrow \{v : u \leq_{J[t]} v\} \in E$, exists as J^t is directed. For each $u \in J^t$ there are $A_u, p_u, w(u)$ such that

- \boxminus_6 (a) $A_u \in E$ and
- (b) $p_u \in \mathcal{S}^{k^*}$
- (c) if $v \in A_u$ then $u \leq_{J[t]} v$ and $p_u = \text{tp}(\pi_{u, v}(\bar{t}^v), \emptyset, I_u)$
- (d) $w(u) \in A_u$.

For $p \in \mathcal{S}^{k*}$ let

- \square_7 (a) $Y_p = \{u \in J^t : p_u = p\}$
- (b) $\bar{s}^{u,v} = \pi_{u,v}^t(\bar{t}^v) \upharpoonright \text{supp}(\mathbf{q}(p_u))$ for $u \in J^t, v \in A_u$
- (c) $\bar{s}^u = \bar{s}^{u,w(u)}$.

So

- \square_8 $\langle Y_p : p \in S^{k*} \rangle$ is a partition of J^t .

Fix $p \in \mathcal{S}^k$ for awhile so for each $u \in Y_p$ and $v \in A_u$ by \square_1 , $\varkappa \upharpoonright (\{u\} \times K_u)$ is multiplication from the left by $g_{\pi_{u,v}^t(\bar{t}^v), \mathbf{q}}^{u, \bar{s}}$ (it was \mathbf{q}^v but we have already agreed that $\mathbf{q}^v = \mathbf{q}$). But $p = \text{tp}_{\text{qf}}(\pi_{u,v}^t(\bar{t}^v), \emptyset, J_u)$ as $u \in Y_p, v \in A_u$ and so by Definition 2.11(4) we know that $g_{\pi_{u,v}^t(\bar{t}^v), \mathbf{q}}^{u, \bar{s}}$ is $g_{\pi_{u,v}^t(\bar{t}^v), \mathbf{q}(p)}^{u, \bar{s}}$.

Now $\mathbf{q}(p) \in \Lambda_{k*}^2$ so $\mathbf{q}(p) = \langle \rho_0^p, \rho_1^p, \dots, \rho_{i(p)-1}^p \rangle$ and recall

$$g_{\pi_{u,v}^t(\bar{t}^v), \mathbf{q}(p)} \text{ is } g_{\bar{t}, \rho_0^p} h_{g_{\bar{t}, \rho_1^p}} G_u^{<0} \dots ;$$

so it depends only on $\bar{t} \upharpoonright \text{supp}(\mathbf{q}(p))$ only.

Now consider any two members v_1, v_2 of A_u (so they are above u) comparing the two expressions for $\varkappa \upharpoonright (\{u\} \times K_u)$ one coming from v^1 the second from v^2 we conclude that $g_{\pi_{u,v_2}^t(\bar{t}^{v_2}), \mathbf{q}(p)} = g_{\pi_{u,v_1}^t(\bar{t}^{v_1}), \mathbf{q}(p)}$. As \mathbf{q} is reduced also $\mathbf{q}(p)$ is p -reduced hence by 2.7(3) we conclude that

- \square_9 if $(p \in \mathcal{S}^{k*}, u \in Y_p \subseteq J^t \text{ and } v_1, v_2 \in A_u \text{ then } \pi_{u,v_2}^t(\bar{t}^{v_1}) \upharpoonright \text{supp}(\mathbf{q}(p))$ is a permutation of $\pi_{u,v_2}^t(\bar{t}^{v_2}) \upharpoonright \text{supp}(\mathbf{q}(p))$
this means

- \square_{10} if $p \in \mathcal{S}^{k*}, u \in J^t$ and $v_1, v_2 \in A_u$ then \bar{s}^{u,v_1} is a permutation of \bar{s}^{u,v_2} .

Hence for each $u \in J^t$

- \square_{11} if $v \in A_u$ then $\bar{s}^{u,v}$ is a permutation of $\bar{s}^u = \bar{s}^{u,w(u)}$.

As there are only finitely many permutations of \bar{s}^{u,v_u} , there are \bar{s}^u, A'_u such that

- \square_{12} for $u \in J^t$:
 - (a) $A'_u \in E$
 - (b) $A'_u \subseteq A_u$
 - (c) $\bar{s}^u = \bar{s}^{u,v}$ for every $v \in A'_u$.

Now

□₁₃ if $p \in \mathcal{S}^k$ and $u_1 \leq_{J[t]} u_2$ are from Y_p then $\pi_{u_1, u_2}^t(\bar{s}^{u_2}) = \bar{s}^{u_2}$.

[Why? As E is an ultrafilter on J^t and $A'_{u_1}, A'_{u_2} \in E$ we can find $v \in A'_{u_1} \cap A'_{u_2}$. So for $\ell = 1, 2$ we have $\bar{s}^{u_\ell} = \pi_{u_\ell, v}^t(t^v) \upharpoonright \text{supp}(\mathbf{q}(p)) = \pi_{u_\ell, v}^t(\bar{t}^v \upharpoonright \text{supp}(\mathbf{q}(p)))$.

As $\pi_{u_1, v}^t = \pi_{u_1, u_0}^t \circ \pi_{u_2, v}^t$ we conclude $\bar{s}^{u_1} = \pi_{u_1, u_2}^t(\bar{s}^{u_2})$ is as required.]

Let $\mathcal{S}' = \{p \in \mathcal{S}^{k^*} : Y_p \text{ is an unbound subset of } J^t\}$, so for some $u_* \in J^t$ we have

□₁₄ $J_{\geq u_*}^t \subseteq \cup \{Y_p : p \in \mathcal{S}'\}$.

Also without loss of generality

□₁₅ $k^* = k_1^* + k_2^*$ and $\{0, \dots, k_1^* - 1\} = \cup \{\text{supp}(\mathbf{q}(p)) : p \in \mathcal{S}'\}$

□₁₆ for $p \in \mathcal{S}'$ and $\ell \in \text{supp}(\mathbf{q}(p))$, so s_ℓ^u is well defined for $u \in Y_p$, there is a unique $t \in J_s$ such that:

$$u \in Y_p \Rightarrow \pi_{u, v_*}^5(t) = s_\ell^u.$$

[Why? By clause (d) of Definition 2.1.]

Next we can find \bar{t} such that

□₁₇ (a) $\bar{t} = \langle t_\ell : \ell < k_1^* \rangle$

(b) if $p \in \mathcal{S}'$ and $\ell \in \text{supp}(\mathbf{q}(p))$ then $t_\ell \in I_{v_*}^5$ is as in □₁₆.

[Why? For $i \in \cup \{\text{supp}(\mathbf{q}(p)) : p \in \mathcal{S}'\}$ use □₁₆, as \mathbf{q} is disjoint (see Definition 2.11(2)) there is no case of “double definition”.]

By clause (d) of Definition 3.1, possibly increasing u_*

□₁₈ $p^* = \text{tp}(\pi_{u, v_*}^5(\bar{t}), \emptyset, I_u^5)$ for every $u \in J_{\geq u_*}^t$

□₁₉ let \mathcal{E} be the following equivalence relation on \mathcal{S}^{k^*} , $p_1 \mathcal{E} p_2 \Leftrightarrow \mathbf{q}(p_1) \mathcal{E}_{k_1^*, p \upharpoonright k_1^*}^1 \mathbf{q}(p_2)$;

note they are actually from $\mathcal{S}^{k_1^*}$ and so “ $\mathcal{E}_{k_1^*, p \upharpoonright k_1^*}^1$ -equivalent” is meaningful, see Definition 2.3(4)

□₂₀ let $\bar{e} = \langle e_u : u \in J_{\geq u_*}^t \rangle$ be defined by $e_u = p_u / E$

□₂₁ $E, \bar{t}, \bar{e}, \langle \pi_{u, w(u)}^t(\bar{t}^{w(u)}) : u \in J_{\geq u_*}^t \rangle$ satisfies the demands (f)(α) – (ζ) from Definition 3.2.

[Why? Check.]

Recall $p^* = \text{tp}(\bar{t}, \emptyset, I_{v_*}^5)$ here so let $\bar{s} \in {}^{(k_2^*)}(I_{v_*}^5)$ be as guaranteed to exist by Definition 3.2. Let $\bar{t}^{v^*} := \bar{t} \hat{\ } \bar{s}$. So possibly increasing $u_* \in J^t$ for some p^* we have

□₂₂ if $u \in J_{\geq u_*}^t$ then $p^* = \text{tp}(\pi_{u, v_*}^5(\bar{t} \hat{\ } \bar{s}), \emptyset, I_u^5) = \text{tp}(\bar{t} \hat{\ } \bar{s}, \emptyset, I_{v_*}^5)$.

Let

- ₂₃ (a) $\varrho^* = \mathbf{q}(p^*)$ so $\varrho^* \in \Lambda_{k_1^*, p^*}^2$ and let $\varrho^* = \langle \rho_\ell : \ell < \ell(*) \rangle$
- (b) $\bar{t}_u = \pi_{u, v_*}^5(\bar{t})$ for $u \in J^t$
- (c) let $z_u = z_{\bar{t}_u, \varrho}^{u, s} \in Z_u^{1, s}$ (see Definition 2.4(5A))
- (d) let $f_u = \partial_{u, z_u}^5 \in F_u^5$; (this is not the same as $f_u^{\varkappa!}$).

Now

- ₂₄ for $u_1 \leq_{J[t]} u_2$ we have $f_{u_1} \subseteq f_{u_2}$.

[Why? Check.]

- ₂₅ $\varkappa_{\bar{f}}$ is a finite product of members of $\{\varkappa_x : x \in Z_{v_*}^5\}$.

[Why? Recall \varkappa_x for $x \in Z_{v_*}^5$ is from \otimes_5 . Now use □₂₃.]

Lastly

- ₂₆ $(\varkappa_{\bar{f}}^{-1})\varkappa \in G^+ = \text{Aut}(M_t)$ is the identity on $P_u^{M_t}$ whenever $u \in J_{\geq u_*}^t$.

[Why? By □₂₄ and our choices.]

- ₂₅ $(\varkappa_{\bar{f}}) \in (G_{u_*}^- \subseteq) G^-$.

[Why? By □₂₅ and the definition of $(G_{u_*}^-$ and) G^- .]

- ₂₈ \varkappa is the product (in G^+) of $\varkappa_{\bar{f}} \in G^-$ and $(\varkappa_{\bar{f}}^{-1})\varkappa \in \langle \{\varkappa_x : x \in Z_{v_*}^5\} \rangle$.

[Why? □₂₅ + □₂₇ this is clear.]

As \varkappa was any a member of G^+ we are done proving \otimes_7 .

- ₈ there is a homomorphism \mathbf{h} from $K_{v_*}^5$ onto G^+/G^- which maps g_x to $\varkappa_x G^-$ for $x \in Z_{v_*}^5$.

[Why? By \otimes_7 there is at most one such homomorphism and if it exists it is onto.

So it is enough to show that for any group term, σ if $K_{v_*}^5$ satisfies $K_{v_*} \models \text{"}\sigma(g_{x_1}, \dots, g_{x_{k-1}}) = e\text{"}$ then $\sigma(\varkappa_{x_0}, \dots, \varkappa_{x_{k-1}}) \in G^-$. Let $\langle t_\ell : \ell < \ell^* \rangle$ list $\cup \{\text{his}(x_\ell) : \ell < k\} \subseteq I_{v_*}^5$ and let $u_* \in J^t$ be such that: if $u_* \leq_{J[t]} u$ and $\ell(1), \ell(2) < \ell^*$ we have $I_{v_*}^5 \models t_{\ell(1)} <_I t_{\ell(2)}$ iff $I_u^t \models \pi_{u, v_*}(t_{\ell(1)}) < \pi_{u, v_*}(t_{\ell(2)})$ and similarly for equality, see clause (d) of Definition 3.1.

Let $t_{u, \ell} = \pi_{u, v_*}(t_\ell)$, $x_{u, \ell} = \hat{\pi}_{u, v_*}(x_\ell)$. By the definition of G^- it is enough to show that: if $u_* \leq_{J[t]} u$ then $K_u \models \text{"}\sigma(g_{x_{u, 0}}, \dots, g_{x_{u, k-1}}) = e_{K_u}\text{"}$. By the analysis in 1.6 and §2 (i.e., twisted product) this should be clear.]

- ₉ \varkappa^* is one to one.

[Why? By part of the analysis as for \circledast_7 .]

By $\circledast_8 + \circledast_9$ we are done. $\square_{3.6}$

3.7 Theorem. *Assume*

- (a) $\aleph_0 < \text{cf}(\theta) \leq \theta \leq \kappa$
- (b) $\mathcal{F}_\alpha \subseteq {}^\alpha\kappa$ for $\alpha < \theta$ has cardinality $\leq \kappa$ (also $\mathcal{F}_\alpha \subseteq {}^\alpha\beta$ for some $\beta < \kappa^+$ is O.K.)
- (c) $\mathcal{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathcal{F}_\alpha \text{ for every } \alpha < \theta\}$
- (d) $\gamma = \text{rk}(\mathcal{F}, <_{J_\theta^{\text{bd}}})$, necessarily $< \infty$ so $< (\kappa^\theta)^+$
- (e) for $f_1, f_2 \in \mathcal{F}$, then $f_1 <_{J_\theta^{\text{bd}}} f_2$ or $f_2 <_{J_\theta^{\text{bd}}} f_1$ or $f_2 =_{J_\theta^{\text{bd}}} f_1$; follows from (f)
- (f) for stationarily many $\delta < \theta$ we have: if $f_1, f_2 \in \mathcal{F}_\delta$, then for some $\alpha < \delta$ we have $\beta \in (\alpha, \delta) \Rightarrow (f_1(\beta) < f_2(\beta) \equiv f_1(\alpha) < f_2(\alpha))$.

Then $\tau_\kappa^{\text{atw}} \geq \tau_\kappa^{\text{nlg}} \geq \tau_\kappa^{\text{nlf}} > \gamma$ (on τ_κ^{nlf} see Definition 0.3(4)).

3.8 Theorem. *We can in Theorem 3.7 weaken clause (f) to*

- (f)' (α) $S \subseteq \theta$ is a stationary set consisting of limit ordinals
- (β) D is a normal filter on θ
- (γ) $S \in D$
- (δ) $\bar{J} = \langle J_\delta : \delta \in S \rangle$
- (ε) J_δ is an ideal on δ extending J_δ^{bd} for $\delta \in S$
- (ζ) if $S' \subseteq S, S' \in D^+$ and $w_\delta \in J_\delta$ for $\delta \in S'$ then $\cup\{\delta \setminus w_\delta : \delta \in S'\}$ contains an end segment of θ
- (η) if $\delta \in S$ and $f_1, f_2 \in \mathcal{F}$ then $f_1 \restriction \delta <_{J_\delta} f_2 \restriction \delta$ or $f_2 \restriction \delta <_{J_\delta} f_1 \restriction \delta$ or $f_1 \restriction \delta =_{J_\delta} f_2 \restriction \delta$

Remark. 1) We can justify (f)' by pcf theory quotation, see below.

2) We should prove that the p.o.w.i.s. being existential holds.

Note that in proving 3.7, 3.8 the main point is the “existential limit”. This proof has affinity to the first step in the elimination of quantifiers in the theory of $(\omega, <)$. For this it is better if $I_\theta = (\mathcal{F}, <_{J_\theta^{\text{bd}}})$ has many cases of existence. Toward this we “padded it” in $(*)_0$ of the proof - take care of successor ($f \in \mathcal{F} \Rightarrow f + 1 \in \mathcal{F}$), have zero ($0_\theta \in \mathcal{F}$) without losing the properties we have.

2) The demand of 3.7 may seem very strong, but by pcf theory it is q natural.

3.9 *Observation.* 1) Theorem 3.8 implies Theorem 3.7.

2) If (a)-(d) of 3.7 holds, then $(f) \Rightarrow (f)'$.

3) If (a)-(d) of 3.7 holds then $(f) \Rightarrow (e)$.

Proof. 1) By 2).

2) Let

$$\begin{aligned} S =: \{ \delta < \theta : \delta \text{ is a limit ordinal and if } f_1, f_2 \in \mathcal{F}_\delta \\ \text{then for some } \alpha < \delta \text{ we have } \beta \in (\alpha, \delta) \Rightarrow \\ (f_1(\beta) < f_2(\beta) \equiv f_1(\alpha) < f_2(\alpha)) \}. \end{aligned}$$

By (f) we know that S is a stationary subset of θ . Let \mathcal{D}_θ be the club filter on θ and $D =: \mathcal{D}_\theta + S$, it is a normal filter on θ and $S \in D$. So sub-clauses $(\alpha), (\beta), (\gamma)$ of $(f)'$ holds.

Let $J_\delta = J_\delta^{\text{bd}}$ for $\delta \in S$ so $\bar{J} = \langle J_\delta : \delta \in S \rangle$ satisfies sub-clauses $(\delta), (\varepsilon)$ of $(f)'$. To prove (ζ) assume $S' \subseteq S, S' \in D^+$ and $w_\delta \in J_\delta$ for $\delta \in S'$. Then $\sup(w_\delta) < \delta$ and S' is a stationary subset of δ hence by Fodor lemma for some $\beta(*) < \theta$ the set $S'' = \{ \delta \in S' : \sup(w_\delta) = \beta(*) \}$ is a stationary subset of θ and so $[\beta(*), \theta)$ is an end segment of θ and is equal to $\cup \{ [\beta(*), \delta) : \delta \in S'' \}$ which is included in $\cup \{ \delta \setminus w_\delta : \delta \in S' \}$, as required in (ζ) from $(f)'$, so sub-clause (ζ) really holds.

To prove sub-clause (η) of clause $(f)'$ note that what it says is what is said in (f) .

3) Should be clear. Given $f_1, f_2 \in \mathcal{F}$; by sub-clause (η) of $(f)'$ for each $\delta \in S$ there are $w_\delta \in J_\delta$ and $\ell_\alpha < 3$ such that $\ell_0 = 0 \wedge \alpha \in \delta \setminus w_\delta \Rightarrow f_1(\alpha) < f_2(\alpha)$ and $\ell_\delta = 1 \wedge \alpha \in \delta \setminus w_\delta \Rightarrow f_1(\alpha) = f_2(\alpha)$ and $\ell_\delta = 2 \wedge \alpha \in \delta \setminus w_\delta \Rightarrow f_1(\alpha) > f_2(\alpha)$. So for some $\ell < 2$ the set $S' := \{ \delta \in S : \ell_\delta = \ell \}$ is stationary, hence $\cup \{ \delta \setminus w_\delta : \delta \in S' \}$ include an end segment of θ and we are easily done. $\square_{3.9}$

Proof of 3.8. Without loss of generality

- (*)₀ (a) $(\forall f \in \mathcal{F})(\exists^\infty g \in \mathcal{F})(f \upharpoonright [1, \theta) = g \upharpoonright [1, \theta))$;
moreover for $f \in \mathcal{F}$ we have
 $\omega = \{ g(0) : g \in \mathcal{F} \text{ and } g \upharpoonright [1, \theta) = f \upharpoonright [1, \theta) \}$
- (b) $\alpha < \beta < \theta \Rightarrow \mathcal{F}_\alpha = \{ f \upharpoonright \alpha : f \in \mathcal{F}_\beta \}$; moreover $\alpha < \theta \Rightarrow \mathcal{F}_\alpha = \{ f \upharpoonright \alpha : f \in \mathcal{F} \}$
- (c) if $f \in \mathcal{F}$, then $f + 1 \in \mathcal{F}$
- (d) the $f \in {}^\theta\{0\}$, the constantly zero function, belongs to \mathcal{F} .

[Why? Let $\mathcal{F}' = \{f \in {}^\theta\kappa : \text{for some } n, (\forall \alpha < \theta)(f(1 + \alpha) = n) \wedge f(0) < \omega \text{ or for some } f' \in \mathcal{F} \text{ and } n < \omega \text{ we have } (\forall \alpha < \theta)(f(1 + \alpha) = \omega(1 + f'(\alpha)) + n) \wedge f(0) < \omega\}$ and for $\alpha < \theta$, replace \mathcal{F}_α by $\mathcal{F}'_\alpha = \{f \upharpoonright \alpha : f \in \mathcal{F}'\}$. Now check that (a)–(e), $(f)'$ of the assumption still holds.]

We define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ as follows:

$$(*)_1 \quad (a) \quad J = (\theta + 1; <)$$

$$(b)(\alpha) \quad \text{let } I_\theta = (\mathcal{F}, <_{J^{\text{bd}}}) \text{ and}$$

$$(\beta) \quad I_\alpha = (\mathcal{F}_{1+\alpha+1}, <_{\alpha+1}) \text{ for } \alpha < \theta \text{ where}$$

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(1 + \alpha) < f_2(1 + \alpha)$$

$$(c) \quad \text{for } \alpha < \beta < \theta + 1 \text{ let } \pi_{\alpha,\beta} : I_\beta \rightarrow I_\alpha \text{ be}$$

$$\pi_{\alpha,\beta}(f) = f \upharpoonright (1 + \alpha + 1).$$

Note that

$$(*)_2 \quad I_\alpha \text{ is a non-trivial (see Definition 1.1(6)).}$$

[Why? By $(*)_0(a)$ and the choice of $<_{I_\alpha}$ in $(*)_1(b)(\beta)$.]

$$(*)_3 \quad \mathfrak{s} = (J, I, \bar{\pi}) \text{ is a p.o.w.i.s. even nice}$$

[Note clause (d) of Definition 3.1 holds by clause (e) of Theorem 3.7.]

$$(*)_4 \quad \mathfrak{s} \text{ is a limit of } \mathfrak{t} =: \mathfrak{s} \upharpoonright \theta = ((\theta, <), \bar{I} \upharpoonright \theta, \bar{\pi} \upharpoonright \theta).$$

[Why? Note that clause (d) of Definition 3.1 holds by clause (f) here and Fodor lemma. Easy to check the other clauses.]

$$(*)_5 \quad \mathfrak{t} \text{ is a } \kappa\text{-p.o.w.i.s.}$$

[Why? Check, as $\alpha < \theta \Rightarrow |\mathcal{F}_\alpha| \leq \kappa$.]

Now $G_\theta^\mathfrak{s}$ is an almost κ -automorphism group by Claim 3.4, the “existential limit” holds by $(*)_6$ below (note: J is linear). Now $\text{rk}(I_\theta^\mathfrak{s}) = \gamma$ and $H_\theta^\mathfrak{s}$ is a subgroup of $G_\theta^\mathfrak{s}$ of cardinality $2 \leq \kappa$.

By 1.8

$$\tau_{G_\theta^\mathfrak{s}, G_\theta^{\leq 1, \mathfrak{s}}}^{\text{nlg}} = \text{rk}(I_\theta^\mathfrak{s}) = \gamma$$

and $\text{nor}_{G_\theta^\mathfrak{s}}^{\leq \infty}(H_\theta^\mathfrak{s}) = G_\theta^\mathfrak{s}$ and by 2.10(4), $\tau_{G_\theta^\mathfrak{s}, H_\theta^\mathfrak{s}}^{\text{nlf}} = \gamma$.

We still have to check

$$(*)_6 \quad \text{“}\mathfrak{s} \text{ is an existential limit of } \mathfrak{t}\text{”, see Definition 3.2.}$$

That is we have to prove clause (f) of 3.2, so we should prove its conclusion, assuming its assumption which means in our case

- \otimes_1 (a) $k = k_1 + k_2$, \mathcal{E} is an equivalence relation on \mathcal{S}^k
- (b) $\bar{f} \in {}^{k_1}(\mathcal{F}_\theta)$ and $\alpha(*) < \delta$
- (c) $\bar{e} = \langle e_\alpha \in [\alpha(*), \theta) \rangle$ is such that $e_\alpha \in \mathcal{S}^k / \mathcal{E}$
- (d) $\langle \bar{g}^\alpha : \alpha \in [\alpha(*), \theta) \rangle$ is such that $\bar{g}^\alpha \in {}^{(k_2)}(\mathcal{F}_\alpha)$
- (e) if $\alpha(*) \leq \alpha < \beta$ then:

e_α is the \mathcal{E} -equivalence class of $\text{tp}_{\text{qf}}(\langle f_\ell(1 + \alpha) : \ell < k_1 \rangle \wedge \langle g_\ell^\beta(1 + \alpha) : \ell < k_2 \rangle, \emptyset, \kappa)$.

Without loss of generality [recalling clause (e) of the assumption and $(*)_0(c)$]

- \otimes_2 (f) $\langle f_\ell : \ell < k_1 \rangle$ is $\leq_{J_\theta^{\text{bd}}}$ -increasing
- (g) f_0 is constantly zero
- (h) for each $\ell < k_1 - 1$ we have: $f_{\ell+1} = f_\ell \bmod J_\theta^{\text{bd}}$ or $f_{\ell+1} = f_\ell + 1 \bmod J_\theta^{\text{bd}}$ or $f_\ell + \omega \leq f_{\ell+1} \bmod J_\omega^{\text{bd}}$
- (i) $\langle f_\ell : \ell < k_1 \rangle$ is without repetition
- (j) $\langle f_\ell(0) : \ell < k_1 \rangle$ is without repetition.

Possibly increasing $\alpha(*) < \theta$ without loss of generality

- \otimes_3 if $\alpha \in [\alpha(*), \theta)$ and $\ell_1, \ell_2 < k_1$ then $f_{\ell_1}(\alpha) < f_{\ell_2}(\alpha) \Leftrightarrow f_{\ell_1}(\alpha(*)) < f_{\ell_2}(\alpha(*))$.

Hence by clause (f) of \otimes_2

- \otimes_4 $\langle f_\ell(\alpha(*)) : \ell < k_1 \rangle$ is non-decreasing.

For notational simplicity

- \otimes_5 (a) $\langle f_e \upharpoonright \delta : \ell < k_1 \rangle = \langle g_\ell^\delta : \ell < k_1 \rangle$ so $k_1 < k_2$
- (b) if $\ell_1 < k_2, \ell_2 \in [k_1, k_2)$ then $g_{\ell_1}^\delta = g_{\ell_2}^\delta \equiv g_{\ell_1}^\delta(0) = g_{\ell_2}^\delta(0)$.

Next for some p^*

- \otimes_6 $p^* \in \mathcal{S}^k$ and for some $S' \subseteq S$ from D^+ , for every $\delta \in S'$ for the J_δ -majority of $\alpha < \delta$, say $\alpha \in \delta \setminus w_\alpha, w_\alpha \in J_\delta$, we have $p^* = \text{tp}_{\text{qf}}(\langle g_\ell^\delta \upharpoonright (1 + \alpha + 1) : \ell < k_2 \rangle, \emptyset, I_\alpha)$.

[Why? By sub-clause (η) of clause $(f)'$, as J_δ is an ideal (applied to $(g_{\ell_1}^\delta, g_{\ell_2}^\delta)$ for every $\ell_1, \ell_2 < k_2$) for each $\delta \in S$ we can choose $w_\delta \in J_\delta$ and $q_\delta \in \mathcal{S}^k$ such that for every $\alpha \in (\delta \setminus w)$ we have $\text{tp}_{\text{qf}}(\langle g_\ell^\delta(1 + \alpha) : \ell < k_2 \rangle, \emptyset, I_\alpha)$ is equal to q_δ . For each $p \in \mathcal{S}^k$ let $S_p = \{\delta \in S : q_\delta = p\}$. So $S = \cup\{S_p : p \in \mathcal{S}^k\}$, hence for some p we have $S_p \in D^+$. So let $S' = S_p, p^* = p$.]

So without loss of generality considering the way I_α was defined by \otimes_5

\otimes_7 there are $E_1^*, E_2^*, <_*$ such that

- (a) E_1^* is an equivalence relation on $k_2 = \{0, \dots, k_2 - 1\}$
- (b) E_2^* is an equivalence relation on k_2 refining E_1^*
- (c) $<_*$ linearly order k_2
- (d) if $\delta \in S', \alpha \in \delta \setminus w_\delta$ so $p^* = \text{tp}_{\text{qf}}(\langle g_\ell^\delta(\alpha) : \ell < k_2 \rangle)$ then:
 - (α) $\ell_1 E_2^* \ell_2$ iff $g_{\ell_1}^\delta(1 + \alpha) = g_{\ell_2}^\delta(1 + \alpha)$
 - (β) $\ell_1 E_2^* \ell_2$ iff $g_{\ell_1}^\delta \upharpoonright (1 + \alpha + 1) = g_{\ell_2}^\delta \upharpoonright (1 + \alpha + 1)$
 - (γ) $(\ell_1/E_1^*) <_* (\ell_2/E_1^*)$ iff $g_{\ell_1}^\delta(1 + \alpha) < g_{\ell_2}^\delta(1 + \alpha)$.

Let $\langle u_0, \dots, u_{m-1} \rangle$ list the E_1^* -equivalence classes in $<_*$ -increasing order. Necessary $0 \in u_0$.

Let $\alpha_* = \min(\delta_* \setminus w_{\delta_*})$ where $\delta_* = \min(S')$. We now define $g_\ell \in {}^\theta \kappa$ for $\ell < k_2$ as follows. So necessarily for a unique $i = i(\ell), \ell \in u_i$ and let $i_1 = i_1(\ell) \leq i$ be maximal such that $u_{i_1} \cap \{0, \dots, k_1 - 1\} \neq \emptyset, j_2 = j_2(\ell) = \min(\{u_1 \cap \{0, \dots, k_1 - 1\}\})$. It is well defined as necessary $0 \in u_0$ because f_0 is constantly zero. Now we let

$$\boxdot_0 \quad g_\ell = (g_\ell^{\alpha_*} \upharpoonright \{0\}) \cup ((f_{j_2} + (i - i_1)) \upharpoonright [1, \theta)).$$

Now

- \boxdot_1 if $\ell < k_1$ then $g_\ell = f_\ell$
[Why? Check the definition $g_\ell^{\alpha_*}(0) = f_\ell(0)$ as $g_\ell^{\alpha_*} = f_\ell$.]
- \boxdot_2 $g_\ell \in \mathcal{F}$ for $\ell < k_2$
[Why? As $f_{j_2} \in \mathcal{F}$ and clauses (a)+(c) of $(*)_0$.]
- \boxdot_3 if $\ell_1 E_2^* \ell_2$ then $g_{\ell_1} = g_{\ell_2}$
[Why? First, as $\ell_1 E_2^* \ell_2$ we have $g_\ell(0) = g_{\ell_1}^{\alpha_*}(0) = g_{\ell_2}^{\alpha_*}(0) = g_\ell(0)$. Second, clearly $i(\ell_1) = i(\ell_2), i_1(\ell_1) = i_1(\ell_2)$ and $j_2(\ell_1) = j_2(\ell_2)$ hence for $\alpha \in [1, \theta)$ we have

$$\begin{aligned} g_{\ell_1}(\alpha) &= (f_{j_2(\ell_1)}(\alpha) + (i(\ell_1) - i_1(\ell_1))) = \\ &= f_{j_2(\ell_1)}(\alpha) + (i(\ell_2) - i_1(\ell_2)) = g_{\ell_2}(\alpha). \end{aligned}$$

So we are done.]

- ₄ if $\ell_1, \ell_2 < k_2$ but $\neg(\ell_1 E_2^* \ell_2)$ then $g_{\ell_1} \neq g_{\ell_2}$
 [Why? If $\ell_1, \ell_2 < k_1$ then $g_{\ell_1} = f_{\ell_1} \neq f_{\ell_2} = g_{\ell_2}$. If $\ell_1 < k_2, \ell_2 \in [k_1, k_2)$ as $\neg(\ell_1 E_1^* \ell_2)$ by $(*)_5(b)$ we have $g_{\ell_1}^{\alpha^*}(0) \neq g_{\ell_2}^{\alpha^*}(0)$, hence $g_{\ell_1}(0) = g_{\ell_1}^{\alpha^*}(0) \neq g_{\ell_2}^{\alpha^*}(0) = g_{\ell_2}(0)$ hence $g_{\ell_1} \neq g_{\ell_2}$. Lastly, if $\ell_1 \in [k_1, k_2), \ell_2 < k_2$ the proof is similar.]
- ₅ if $\ell_1, \ell_2 < k_2, \ell_1 E_1^* \ell_2$ then $\neg(g_{\ell_1} <_{I_\theta} g_{\ell_2})$
 [Why? As $g_{\ell_1} \upharpoonright [1, \theta) = g_{\ell_2} \upharpoonright [1, \theta)$, so $g_{\ell_1} = g_{\ell_2} \bmod J_\theta^{\text{bd}}$, so $I_\theta \models \neg(g_{\ell_1} < g_{\ell_2})$.]
- ₆ if $\ell_1, \ell_2 < k_2$ and $(\ell_1/E_1^*) <_* (\ell_2/E_2^*)$ then $g_{\ell_1} <_{I_\theta} g_{\ell_2}$
 [Why? If $f_{j_2(\ell_1)} + \omega \leq f_{j_2(\ell_2)} \bmod J_\theta^{\text{bd}}$ then easily $g_{\ell_1} <_{J_\theta^{\text{bd}}} f_{j_2(\ell_1)} + \omega \leq_{J_\theta^{\text{bd}}} f_{j_2(\ell_2)} \leq_{J_\theta^{\text{bd}}} g_{\ell_2}$ so we are done. If $j_2(\ell_1) = j_2(\ell_2)$ then as still $i(\ell_1) < i(\ell_2)$ we have $g_{\ell_1} =_{J_\theta^{\text{bd}}} f_{j_2(\ell_1)} + (i(\ell_1) - j_2(\ell_1)) < f_{j_2(\ell_1)} + (i(\ell_2) - j_2(\ell_1)) =_{J_\theta^{\text{bd}}} g_{\ell_2}$ as required. If $j_2(\ell_1) \neq j_2(\ell_2)$ then necessarily $j_2(\ell_1) < j_2(\ell_2), i_1(\ell_1) < i_1(\ell_2)$ moreover $i_1(\ell_1) \leq i(\ell_1) < j_2(\ell_2) \leq i(\ell_2)$ but by $\textcircled{*}(h)$ we have $f_{j_1(\ell_1)} + (j_2(\ell_1) - i_1(\ell_1)) \leq_{J_\theta^{\text{bd}}} f_{j_2(\ell_2)}$ so we are easily done.]

Together $\langle g_\ell : \ell < k_2 \rangle$ is as required for proving $(f)'$ of 3.2, the definition of existential limit, i.e. $(*)_6$. □_{3.7} □_{3.8}

We quote

3.10 Claim. Assume $\text{cf}(\kappa) = \theta > \aleph_0, \alpha < \kappa \Rightarrow (\alpha)^\theta < \kappa$ and $\lambda = \kappa^\theta$. Then we can find $\langle \mathcal{F}_i : i \leq \theta \rangle, S, D$ satisfying the conditions from 3.8 with $\gamma = \lambda$ (and more).

Proof. By 3.11 and [Sh:g]. □_{3.10}

3.11 Claim. Assume

- ⊛ (a) $\bar{\lambda} = \langle \lambda_i : i < \theta \rangle$ is an increasing sequence of regular cardinals with limit κ
- (b) $\lambda = \text{tcf}(\prod_{i < \theta} \lambda_i, <_{J_\theta^{\text{bd}}})$
- (c) $\max \text{pcf}\{\lambda_i : i < j_*\} < \kappa$ for every $j < \theta$.

1) Then there are D, S^*, u such that

- (α) $u \in [\theta]^\theta, S^* \subseteq \theta$ is stationary

- (β) there are no $\zeta < \theta, u_\varepsilon \in [u]^\theta$ for $\varepsilon < \theta$ such that for a club of $\delta < \theta$ if $\delta \in S^*$ then for at least one $\varepsilon < \delta$ we have $\max \text{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} < \max \text{pcf}\{\lambda_i : i \in \delta\}$ hence
- (γ) D is a normal filter on θ where: D is $\{S \subseteq \theta : \text{for some sequence } \langle u_\varepsilon : \varepsilon < \theta \rangle \text{ of subsets of } \theta \text{ each of cardinality } \theta \text{ and for some club } E \text{ of } \theta, \text{ if } \delta \in E \cap S \cap S^* \text{ then for every } \varepsilon < \delta \text{ we have } \max \text{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} = \max \text{pcf}\{\lambda_i : i \in \delta \cap u\}\}$
- (δ) by renaming $u = \theta$ and for $\delta \in S^*$ let $J_\delta = \{u \subseteq \delta : \max \text{pcf}\{\lambda_i : i \in \delta \setminus u\} < \max \text{pcf}\{\lambda_i : i < \delta\}\}$.

2) We can choose $\mathcal{F}_i \subseteq \prod_{j < i} \lambda_j$ for $i \leq \theta$ such that all the conditions in 3.8 holds.

Proof. By [Sh:g, II,3.5], see on this [Sh:E12, §18].

3.12 Conclusion. If κ is strong limit singular of uncountable cofinality then $\tau_\kappa^{\text{atw}} \geq \tau_\kappa^{\text{nlg}} \geq \tau_\kappa^{\text{nlf}} > 2^\kappa$.

Proof. By 3.8 and 3.11. □_{3.12}

3.13 Remark. 1) If $\kappa = \kappa^{\aleph_0}$ do we have $\tau_\kappa^{\text{atw}} \geq \tau_\kappa^{\text{nlg}} \geq \tau_\kappa^{\text{nlf}} > \kappa^+$? But if $\kappa = \kappa^{<\kappa}$ \aleph_0 then quite easily yes.

2) In 3.12 we can weaken “ κ is strong limit”. E.g. if κ has uncountable cofinality and $\alpha < \kappa \Rightarrow |\alpha|^{\text{cf}(\kappa)} < \kappa$, then $\tau_\kappa^{\text{nlf}} > \kappa^{\text{cf}(\kappa)}$; see more in [Sh:E12, §18].

3) We elsewhere will weaken the assumption in 3.7, 3.8 but deduce only that τ_κ^{nlg} is large.

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